COMPARISON OF HIGHER INVARIANTS FOR SPECTRAL TRIPLES

J. KAAD

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen, Denmark

Abstract

In this paper we prove that the multiplicative character of A. Connes and M. Karoubi and the determinant invariant of L. G. Brown, J. W. Helton and R. E. Howe agree up to a canonical homomorphism.

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email: `kaad@math.ku.dk`.

1. Introduction

In their paper, [10], A. Connes and M. Karoubi define a multiplicative character on algebraic K-theory

$$M_F : K_{2p}(\mathcal{A}) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

for each odd Fredholm module (F, H) satisfying the additional growth condition

$$[F, a] \in \mathcal{L}^{2p}(H) \quad \text{for all} \quad a \in \mathcal{A} \tag{1}$$

By the work of A. L. Carey, J. Phillips and F. A. Sukochev, [11] and [12], each odd unital spectral triple $(\mathcal{A}, H, \mathcal{D})$ with resolvent

$$(\lambda - \mathcal{D})^{-1} \in \mathcal{L}^q(H) \quad \text{for each} \quad \lambda \in \mathbb{C} - \mathbb{R}$$

in the q^{th} Schatten ideal for some $1 \leq q < 2p$, gives rise to an odd $2p$-summable Fredholm module in a canonical way. The Fredholm operator $F \in \mathcal{L}(H)$ being the sign operator

$$F(x) = \begin{cases} x & \text{for} \quad x \in 1_{[0,\infty)}(\mathcal{D}) \\ -x & \text{for} \quad x \in 1_{(-\infty,0)}(\mathcal{D}) \end{cases}$$

We can therefore associate a multiplicative character

$$M_{\mathcal{D}} : K_{2p}(\mathcal{A}) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

to each odd q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ with $1 \leq q < 2p$.

In a different direction, L.G. Brown has defined a determinant invariant

$$\text{Det}_X : K_2(\mathcal{B}) \to \mathbb{C}^*$$

for each exact sequence of $\mathbb{C}$-algebras

$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{E} \xrightarrow{s} \mathcal{B} \longrightarrow 0$$

equipped with an injective homomorphism $\iota : \mathcal{E} \to \mathcal{L}(H)$ such that

$$(\iota \circ i)(T) = T \quad \text{for all} \quad T \in \mathcal{L}^1(H) \tag{2}$$

See the papers [4] and [5].

In the case where $\mathcal{B}$ is commutative this invariant is related to the work of J. W. Helton and R. E. Howe on traces of commutators, [15], via the identity

$$\text{Det}_X\{s(S), s(T)\} = \exp(\text{Tr}[S, T])$$

Here $\{s(S), s(T)\} \in K_2(\mathcal{B})$ denotes the Steinberg symbol.

Given an odd q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ with $1 \leq q < 2$ we construct an exact sequence

$$X_{\mathcal{D}} : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{E} \xrightarrow{s} \mathcal{B} \longrightarrow 0 \tag{3}$$

an injective homomorphism $\iota : \mathcal{E} \to \mathcal{L}(H)$ satisfying (2) and a surjective homomorphism

$$R : \mathcal{A} \to \mathcal{B}$$

In particular the spectral triple $(\mathcal{A}, H, \mathcal{D})$ furnishes both a multiplicative character

$$M_{\mathcal{D}} : K_2(\mathcal{A}) \to \mathbb{C}/(2\pi i)\mathbb{Z}$$

a determinant invariant

$$\mathrm{Det}_{X_{\mathcal{D}}} : K_2(\mathcal{B}) \to \mathbb{C}^*$$

and a canonical link between them

$$R_* : K_2(\mathcal{A}) \to K_2(\mathcal{B})$$

The main result of this paper can then be expressed as the commutativity of the diagram

$$\begin{array}{ccc} K_2(\mathcal{A}) & \xrightarrow{R_*} & K_2(\mathcal{B}) \\ {\scriptstyle M_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Det}_{X_{\mathcal{D}}}} \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow[\exp]{} & \mathbb{C}^* \end{array} \tag{4}$$

The multiplicative character and the determinant invariant thus agree upto the homomorphism $R_* : K_2(\mathcal{A}) \to K_2(\mathcal{B})$.

This should be seen in relation with a result of A. Connes and M. Karoubi describing the multiplicative character in terms of a central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \Gamma \longrightarrow E(\mathcal{M}^1) \longrightarrow 1$$

Indeed some of the main ideas for our proof are taken from [10, Paragraphe 5].

One important advantage of comparing the multiplicative character and the determinant invariant is that the latter admits calculation. As an example we look at the spectral triple $\big(C^\infty(S^1), L^2(S^1), -i\frac{d}{dt}\big)$. For any smooth loop $a : S^1 \to \mathbb{C}$ we prove that

$$M_{\mathcal{D}}\{z, e^a\} = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)d\theta \in \mathbb{C}/(2\pi i)\mathbb{Z}$$

The algebraic K-group $K_2\big(C^\infty(S^1)\big)$ thus contains information about the 0^{th} Fourier coefficient of a. In this context we should also mention the work in [28] on a central extension of $E\big(C^\infty(S^1)\big)$ by $\mathbb{C}^*$.

The paper is organized as follows.

In the first section we give an alternative description of the universal determinant invariant

$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

The main ingredient is a different construction of the homomorphism

$$f_\varphi : H_2(H) \to \mathrm{Ker}(\varphi)/[G, \mathrm{Ker}(\varphi)]$$

induced by a surjective group homomorphism $\varphi : G \to H$. This approach uses a cokernel complex as introduced by M. Levine in [19]. What is gained at first sight is a homomorphism which is more suitable for calculations since it is defined without the use of set theoretical sections. On a deeper level, the alternative definition of the determinant invariant which arises seems to be crucial for our main proof to work.

In section 3 we review the definition of the universal odd multiplicative character

$$M_U : K_{2p}(\mathcal{M}^1) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

introducing the relevant concepts as we need them. In particular we describe the relative Chern character

$$\mathrm{ch}_n^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to HC_{n-1}^{\mathrm{cont}}(A)$$

in details. To this end we introduce a homology theory, $H_n\big(A_0(\Delta)\big)$, of C^2 maps

$$\sigma : \Delta^n \to GL(A) \qquad \sigma(\mathbf{0}) = 1$$

as well as the logarithm

$$\gamma : H_n\big(A_0(\Delta)\big) \to HC_{n-1}^{\mathrm{cont}}(A)$$

given by

$$\gamma(\sigma) = \frac{(-1)^n}{n!} \sum_{s \in \Sigma_n} \mathrm{sgn}(s) \int_{\Delta^n} \frac{\partial \sigma}{\partial t_{s(1)}} \cdot \sigma^{-1} \otimes \ldots \otimes \frac{\partial \sigma}{\partial t_{s(n)}} \cdot \sigma^{-1} dt_1 \wedge \ldots \wedge dt_n$$

We remark that for each C^2 map

$$\sigma : \Delta^1 \to \mathcal{G} \qquad \sigma(\mathbf{0}) = 1$$

where $\mathcal{G}$ denotes the operators of determinant class we have

$$(\exp \circ \mathrm{Tr} \circ \gamma)(\sigma) = \det\big(\sigma(\mathbf{1})\big)$$

giving a hint of the connection between determinants and the multiplicative character. Another important observation, which we present in section 4, is the factorization of the homomorphism

$$\tau_1 : HC_1^{\mathrm{cont}}(\mathcal{M}^1) \to \mathbb{C}$$

induced by the cyclic cocycle of A. Connes through the relative homology group $HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1)$

$$\begin{tikzcd} HC_1^{\mathrm{cont}}(\mathcal{M}^1) \arrow[r, "\partial"] \arrow[d, "\tau_1"'] & HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1) \arrow[dl, "T"] \\ \mathbb{C} & \end{tikzcd}$$

Together with considerations on the relative normalized singular homology group

$$H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big)$$

and its relations to relative continuous cyclic homology these observations lead to a proof of the main theorem: The universal multiplicative character agrees with the universal determinant invariant up to a canonical homomorphism

$$\begin{tikzcd} K_2(\mathcal{M}^1) \arrow[r, "R_*"] \arrow[d, "M_U"'] & K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \arrow[d, "\mathrm{Det}_U"] \\ \mathbb{C}/(2\pi i)\mathbb{Z} \arrow[r, "\exp"'] & \mathbb{C}^* \end{tikzcd}$$

In section 5 we review the transition from spectral triples to Fredholm modules and construct the associated exact sequence (3). The commutativity of the diagram (4) is then deduced as a corollary to the main theorem.

The paper concludes with a calculation of the multiplicative character in a concrete example.

Acknowledgements We would like to thank Ryszard Nest for his support and helpful comments.

2. The Determinant Invariant

2.1. **Preliminaries on Relative Group Homology.** In this section we will review some standard homological constructions. This is needed in order to give precise descriptions of the isomorphisms in Theorem 2.1 and Theorem 2.2 which we will need throughout the paper.

The group homology with integer coefficients of a group G is the homology of the chain complex $\big(\mathbb{Z}[G^n], d\big)$ where the boundary

$$d : \mathbb{Z}[G^n] \to \mathbb{Z}[G^{n-1}] \qquad d = \textstyle\sum_{i=0}^{n}(-1)^i d_i$$

is given by the face operators

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{for} \quad i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{for} \quad i \in \{1, 2, \ldots, n-1\} \\ (g_1, \ldots, g_{n-1}) & \text{for} \quad i = n \end{cases} \tag{5}$$

In degree $n \in \{1, 2, \ldots\}$ we denote the cycles by $Z_n(G)$, the boundaries by $B_n(G)$ and the homology by $H_n(G)$.

Each group homomorphism $\varphi : G \to H$ induces a group homomorphism on homology $\varphi_* : H_n(G) \to H_n(H)$ which on chains are extended linearly from

$$\varphi_*(g_1, \ldots, g_n) = \big(\varphi(g_1), \ldots, \varphi(g_n)\big)$$

Explaining the functoriality of group homology.

The homology groups $H_n(G)$ and $H_n(H)$ fit in a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "\varphi_*"] & H_{n+1}(H) \arrow[r, "\partial"] & H_n(G,H) \arrow[r, "p"] & H_n(G) \arrow[d, "\varphi_*"] \\ \cdots & H_{n-1}(G) \arrow[l, "\varphi_*"] & H_{n-1}(G,H) \arrow[l, "p"] & H_n(H) \arrow[l, "\partial"] \end{tikzcd} \tag{6}$$

which terminates at $H_0(G,H)$. The relative group homology, $H(G,H)$, is the homology of the totalized shifted cone complex $\big(\mathrm{Tot}_n(\mathrm{Cone}(\varphi)[-1]), d(\varphi)\big)$ where the boundary

$$d(\varphi) : \mathbb{Z}[H^{n+1}] \oplus \mathbb{Z}[G^n] \to \mathbb{Z}[H^n] \oplus \mathbb{Z}[G^{n-1}]$$

is given by

$$d(\varphi)(y, x) = \big(dy + \varphi_*(x), -dx\big) \qquad (y, x) \in \mathbb{Z}[H^{n+1}] \oplus \mathbb{Z}[G^n]$$

The map $p : H_n(G,H) \to H_n(G)$ is induced by the projection $p(y,x) = x$. The map $\partial : H_n(H) \to H_{n-1}(G,H)$ is induced by the inclusion $\partial(y) = (y, 0)$.

Each commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & H \arrow[d, "\beta"] \\ G' \arrow[r, "\varphi'"'] & H' \end{tikzcd} \tag{7}$$

of group homomorphisms induces a group homomorphism $(\beta,\alpha)_* : H_n(G,H) \to H_n(G',H')$ which on chains is given by

$$(\beta,\alpha)_*(y,x) = \big(\beta_*(y),\alpha_*(x)\big) \qquad (y,x) \in \mathbb{Z}[H^{n+1}] \oplus \mathbb{Z}[G^n]$$

Explaining the functoriality of the relative group homology.

For each $n \in \{1,2,\ldots\}$ let $G^n \times_{H^n} G^n$ denote the set of pairs $(g^1,g^2) \in G^n \times G^n$ which coincide in H^n

$$\varphi_*(g^1) = \varphi_*(g^2) \in H^n$$

Let $\mathrm{Coker}_n(\delta_\varphi)$ denote the abelian group $\mathbb{Z}[G^n \times_{H^n} G^n]$ modulo the relation

$$(g^1,g^2) + (g^2,g^3) \sim (g^1,g^3) \qquad (g^1,g^2)\,,(g^2,g^3) \in G^n \times_{H^n} G^n$$

We get a chain complex $\big(\mathrm{Coker}_n(\delta_\varphi), d\big)$ with boundary

$$d : \mathrm{Coker}_n(\delta_\varphi) \to \mathrm{Coker}_{n-1}(\delta_\varphi)$$

extending linearly from

$$(g^1,g^2) \mapsto \textstyle\sum_{i=0}^n (-1)^i \big(d_i(g^1), d_i(g^2)\big) \qquad (g^1,g^2) \in G^n \times_{H^n} G^n$$

The homology groups $H_n\big(\mathrm{Coker}(\delta_\varphi)\big)$ are related to the relative homology groups by the map

$$i \circ \varepsilon : \mathrm{Coker}_n(\delta_\varphi) \to \mathrm{Tot}_n\big(\mathrm{Cone}(\varphi)[-1]\big)$$

extending linearly from

$$(i \circ \varepsilon)(g^1,g^2) = (0, g^1 - g^2) \qquad (g^1,g^2) \in G^n \times_{H^n} G^n$$

Each commutative diagram of the form (7) induces a group homomorphism

$$\alpha_* : H_n\big(\mathrm{Coker}(\delta_\varphi)\big) \to H_n\big(\mathrm{Coker}(\delta_{\varphi'})\big)$$

which on chains extends linearly from

$$\alpha_*(g^1,g^2) = \big(\alpha_*(g^1), \alpha_*(g^2)\big)$$

We then get the commutative diagram

$$\begin{tikzcd} H_{n+1}(H) \arrow[r, "\partial"] \arrow[d, "\beta_*"'] & H_n(G,H) \arrow[d, "(\beta,\alpha)_*"] & H_n\big(\mathrm{Coker}(\delta_\varphi)\big) \arrow[l, "i\circ\varepsilon"'] \arrow[d, "\alpha_*"] \\ H_{n+1}(H') \arrow[r, "\partial"'] & H_n(G',H') & H_n\big(\mathrm{Coker}(\delta_{\varphi'})\big) \arrow[l, "i\circ\varepsilon"] \end{tikzcd} \tag{8}$$

We cite the following theorems from [19].

Theorem 2.1. *Suppose that $\varphi : G \to H$ is surjective. Then the map*
$$i \circ \varepsilon : \mathrm{Coker}_*(\delta_\varphi) \to \mathrm{Tot}_*\big(\mathrm{Cone}(\varphi)[-1]\big)$$
induces an isomorphism on homology
$$i \circ \varepsilon : H_n\big(\mathrm{Coker}(\delta_\varphi)\big) \to H_n(G, H)$$

Let K_φ denote the kernel of $\varphi : G \to H$ and let Γ_φ denote the smallest normal subgroup of G containing all the commutators
$$[g, k] = gkg^{-1}k^{-1} \qquad g \in G \quad \text{and} \quad k \in K$$

Theorem 2.2. *The homomorphism*
$$\psi : \mathbb{Z}[G \times_H G] \to K_\varphi/\Gamma_\varphi \qquad \textstyle\sum_{i=1}^n z_i(x_i, y_i) \mapsto \prod_{i=1}^n (x_i y_i^{-1})^{z_i}$$
descends to an isomorphism
$$\psi : H_1\big(\mathrm{Coker}(\delta_\varphi)\big) \to K_\varphi/\Gamma_\varphi$$

2.2. **The Universal Determinant Invariant.** In this section we will apply the results of Theorem 2.1 and Theorem 2.2 to define a universal determinant invariant
$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$
This will allows us to associate a determinant invariant
$$\mathrm{Det}_X : K_2(\mathcal{B}) \to \mathbb{C}^*$$
to each exact sequence of $\mathbb{C}$-algebras
$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ s\ } \mathcal{B} \longrightarrow 0$$
equipped with an injective homomorphism $\iota : \mathcal{E} \to \mathcal{L}(H)$ satisfying
$$(\iota \circ i)(T) = T \quad \text{for all} \quad T \in \mathcal{L}^1(H) \tag{9}$$

We consider the group homomorphism
$$GL(q) : GL\big(\mathcal{L}(H)\big) \to GL\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$
induced by the quotient map
$$q : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{L}^1(H)$$
Let $I_{GL(q)}$ denote the image of $GL(q)$. Theorem 2.1 and Theorem 2.2 then furnish us with an isomorphism
$$\psi \circ (i \circ \varepsilon)^{-1} : H_1\big(GL(\mathcal{L}(H)), I_{GL(q)}\big) \to K_{GL(q)}/\Gamma_{GL(q)}$$
Since each element $k \in K_{GL(q)}$ satisfies $GL(q)(k) = 1$ the Fredholm determinant yields a homomorphism
$$\det : K_{GL(q)} \to \mathbb{C}^*$$
It descends to the quotient
$$\det : K_{GL(q)}/\Gamma_{GL(q)} \to \mathbb{C}^*$$
since for every $g, l \in GL\big(\mathcal{L}(H)\big)$ and $k \in K_{GL(q)}$ we have
$$\det(lgkg^{-1}k^{-1}l^{-1}) = \det(gkg^{-1}k^{-1}) = \det(gkg^{-1})\det(k^{-1}) = 1$$

For details on the Fredholm determinant we refer to [29] or [31].

In the following we might suppress parts of the isomorphism $\psi \circ (i \circ \varepsilon)^{-1}$. We hope that this will not cause any confusion.

Definition 2.3. *The* universal determinant invariant

$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

is defined as the composition

$$\mathrm{Det}_U = \det \circ \partial \circ i_* \circ h_2$$

of the Hurewicz homomorphism

$$h_2 : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \to H_2(I_{GL(q)})$$

the boundary map on group homology

$$\partial : H_2(I_{GL(q)}) \to H_1\big(GL(\mathcal{L}(H)), I_{GL(q)}\big)$$

and the Fredholm determinant

$$\det : H_1\big(GL(\mathcal{L}(H)), I_{GL(q)}\big) \to \mathbb{C}^*$$

Remark 2.4. *By [23] the first algebraic K-group of $\mathcal{L}(H)$ vanishes. We therefore have the equality $GL\big(\mathcal{L}(H)\big) = E\big(\mathcal{L}(H)\big)$ and by consequence*

$$I_{GL(q)} = E\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

This explains the unconventional choice of range of the Hurewicz homomorphism in the above definition.

Suppose that we have an exact sequence of $\mathbb{C}$-algebras

$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ s\ } \mathcal{B} \longrightarrow 1$$

equipped with an injective homomorphism $\iota : \mathcal{E} \to \mathcal{L}(H)$ satisfying (9). We then have an induced homomorphism

$$\iota : \mathcal{B} \to \mathcal{L}(H)/\mathcal{L}^1(H)$$

which by functoriality of algebraic K-theory yields a homomorphism

$$\iota_* : K_2(\mathcal{B}) \to K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

Definition 2.5. *The determinant invariant*

$$\mathrm{Det}_X : K_2(\mathcal{B}) \to \mathbb{C}^*$$

associated with the exact sequence

$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ \pi\ } \mathcal{B} \longrightarrow 0$$

is defined as the composition

$$\mathrm{Det}_X = \mathrm{Det}_U \circ \iota_*$$

of the homomorphism

$$\iota_* : K_2(\mathcal{B}) \to K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

and the universal determinant invariant

$$\mathrm{Det}_U : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \to \mathbb{C}^*$$

2.3. **Comparing Definitions of The Determinant Invariant.** In this subsection we relate the determinant invariant of Definition 2.5 to the determinant invariant of L. G. Brown, J.W. Helton, R. E. Howe studied and introduced in the papers [4, 5, 15].

Consider the exact sequence of groups

$$1 \longrightarrow K_{GL(q)} \xrightarrow{i} GL\big(\mathcal{L}(H)\big) \xrightarrow{GL(q)} I_{GL(q)} \longrightarrow 1$$

We let $\Gamma_{GL(q)}$ denote the smallest subgroup of $GL\big(\mathcal{L}(H)\big)$ containing the commutators

$$gkg^{-1}k^{-1} \in \Gamma_{GL(q)} \quad \text{for all} \quad g,k \in GL\big(\mathcal{L}(H)\big) \text{ with } GL(q)(k) = 1$$

Passing to the quotient we get a central extension

$$1 \longrightarrow K_{GL(q)}/\Gamma_{GL(q)} \xrightarrow{i} GL\big(\mathcal{L}(H)\big)/\Gamma_{GL(q)} \xrightarrow{GL(q)} I_{GL(q)} \longrightarrow 1 \tag{10}$$

and by consequence a map

$$f_{GL(q)} : H_2(I_{GL(q)}) \to K_{GL(q)}/\Gamma_{GL(q)}$$

See [27] for example. Composing with the Fredholm determinant

$$\det : K_{GL(q)}/\Gamma_{GL(q)} \to \mathbb{C}^*$$

we get a map

$$\det \circ f_{GL(q)} : H_2(I_{GL(q)}) \to \mathbb{C}^*$$

This is the main ingredient of the determinant invariant in the definition of [5].

Relating the two alternative definitions is now a question of showing that the homomorphism

$$f_{GL(q)} : H_2(I_{GL(q)}) \to K_{GL(q)}/\Gamma_{GL(q)}$$

coincides with the composition

$$\psi \circ (i \circ \varepsilon)^{-1} \circ \partial : H_2(I_{GL(q)}) \to K_{GL(q)}/\Gamma_{GL(q)}$$

See Section 2.2.

We undertake a slightly more general task.

Theorem 2.6. *Let $\varphi : G \to H$ be a surjective homomorphism. The homomorphism*

$$f_\varphi : H_2(H) \to K_\varphi/\Gamma_\varphi$$

induced by the central extension

$$1 \longrightarrow K_\varphi/\Gamma_\varphi \xrightarrow{i} G/\Gamma_\varphi \xrightarrow{\varphi} H \longrightarrow 1$$

coincides with the composition

$$\psi \circ (i \circ \varepsilon)^{-1} \circ \partial : H_2(H) \to K_\varphi/\Gamma_\varphi$$

of the group homological boundary map

$$\partial : H_2(H) \to H_1(G,H)$$

and the isomorphism

$$\psi \circ (i \circ \varepsilon)^{-1} : H_1(G,H) \to K_\varphi/\Gamma_\varphi$$

of Theorem 2.1 and Theorem 2.2.

Proof. Let there be given a cycle

$$x = \sum_{i=1}^{2n} (-1)^i (x_i, y_i) \in Z_2(H) \tag{11}$$

representing a class $\mathbf{x} \in H_2(H)$. Choose a set theoretical section

$$t : H \to G \qquad \varphi \circ t = \mathrm{Id}_H$$

of the surjective group homomorphism $\varphi : G \to H$. Recall that the homomorphism

$$f_\varphi : H_2(H) \to K_\varphi / \Gamma_\varphi$$

is given by

$$f_\varphi(\mathbf{x}) = \Big(\prod_{k=1}^n t(x_{2k}) t(y_{2k}) \big(t(x_{2k} y_{2k}) \big)^{-1} \Big) \cdot \Big(\prod_{k=1}^n t(x_{2k-1}) t(y_{2k-1}) \big(t(x_{2k-1} y_{2k-1}) \big)^{-1} \Big)^{-1} \tag{12}$$

For each $k \in \{1, \ldots, n\}$ we set

$$z_k := x_{2k} \quad z_{n+k} := y_{2k} \quad z_{2n+k} := x_{2k-1} y_{2k-1}$$

Likewise for each $l \in \{1, \ldots, n\}$ we set

$$w_l := x_{2l-1} \quad w_{n+l} := y_{2l-1} \quad w_{2n+l} = x_{2l} y_{2l}$$

Note that we then have

$$z_{2n+k} = w_k w_{n+k} \quad \text{and} \quad w_{2n+l} = z_l z_{n+l}$$

Since the boundary

$$d(x) = \sum_{i=1}^{2n} (-1)^i (x_i + y_i - x_i y_i) = \sum_{k=1}^n (z_k + z_{n+k} + z_{2n+k}) - \sum_{l=1}^n (w_l + w_{n+l} + w_{2n+l})$$

vanishes in $\mathbb{Z}[H]$ we can find a bijection

$$\sigma : \{1, \ldots, 3n\} \to \{1, \ldots, 3n\}$$

such that

$$z_j = w_{\sigma(j)} \quad \text{for all} \quad j \in \{1, \ldots, 3n\}$$

The boundary map

$$\partial : H_2(H) \to H_1(G, H)$$

when applied to $\mathbf{x} \in H_2(H)$ is represented by the cycle

$$\Big(0, -d\big(t(x)\big)\Big) \in Z_1\Big(\mathrm{Tot}\big(\mathrm{Cone}(\varphi)[-1]\big)\Big)$$

Where $t(x) \in C_2(G)$ denotes the chain

$$t(x) = \sum_{i=1}^{2n} (-1)^i \big(t(x_i), t(y_i) \big)$$

We then have

$$\begin{aligned}-d\big(t(x)\big) &= -\sum_{i=1}^{2n}(-1)^i\big(t(x_i)+t(y_i)-t(x_i)t(y_i)\big)\\ &= \sum_{l=1}^{n}\big(t(w_l)+t(w_{n+l})-t(w_l)t(w_{n+l})\big)\\ &\qquad -\sum_{k=1}^{n}\big(t(z_k)+t(z_{n+k})-t(z_k)t(z_{n+k})\big)\end{aligned}$$

Using the bijection

$$\sigma:\{1,\dots,3n\}\to\{1,\dots,3n\}$$

we can erase some of the terms and reorder the rest, thus

$$\begin{aligned}-d\big(t(x)\big) = &\sum_{\sigma(2n+k)\in\{1,\dots,2n\}}\big(t(w_{\sigma(2n+k)})-t(w_k)t(w_{n+k})\big)\\ &+\sum_{\sigma(2n+k)\in\{2n+1,\dots,3n\}}\big(t(z_{\sigma(2n+k)-2n})t(z_{\sigma(2n+k)-n})-t(w_k)t(w_{n+k})\big)\\ &+\sum_{\sigma^{-1}(2n+l)\in\{1,\dots,2n\}}\big(t(z_l)t(z_{n+l})-t(z_{\sigma^{-1}(2n+l)})\big)\end{aligned}$$

Recalling that

$$z_{\sigma(2n+k)-2n}z_{\sigma(2n+k)-n}=w_{\sigma(2n+k)}=z_{2n+k}=w_kw_{n+k}\quad\text{and}\quad z_{\sigma^{-1}(2n+l)}=w_{2n+l}=z_lz_{n+l}$$

we get that

$$\begin{aligned}-d\big(t(x)\big) = &\sum_{\sigma(2n+k)\in\{1,\dots,2n\}}\big(t(w_kw_{n+k})-t(w_k)t(w_{n+k})\big)\\ &+\sum_{\sigma(2n+k)\in\{2n+1,\dots,3n\}}\big(t(w_kw_{n+k})-t(w_k)t(w_{n+k})\big)\\ &+\sum_{\sigma(2n+k)\in\{2n+1,\dots,3n\}}\big(t(z_{\sigma(2n+k)-2n})t(z_{\sigma(2n+k)-n})-t(z_{\sigma(2n+k)-2n}z_{\sigma(2n+k)-n})\big)\\ &+\sum_{\sigma^{-1}(2n+l)\in\{1,\dots,2n\}}\big(t(z_l)t(z_{n+l})-t(z_lz_{n+l})\big)\\ = &\sum_{k=1}^{n}\big(t(w_kw_{n+k})-t(w_k)t(w_{n+k})\big)-\sum_{l=1}^{n}\big(t(z_lz_{n+l})-t(z_l)t(z_{n+l})\big)\\ = &\sum_{i=1}^{2n}(-1)^i\big(t(x_i)t(y_i)-t(x_iy_i)\big)\end{aligned}$$

Since we indeed have

$$\varphi\big(t(x_i)t(y_i)\big)-\varphi\big(t(x_iy_i)\big)=0$$

the element $(\psi \circ (i \circ \varepsilon)^{-1} \circ \partial)(\mathbf{x})$ is represented by

$$\Big(\prod_{k=1}^{n} \big(t(x_{2k}) t(y_{2k}) \big(t(x_{2k} y_{2k})\big)^{-1}\big) \cdot \Big(\prod_{k=1}^{n} t(x_{2k-1}) t(y_{2k-1}) \big(t(x_{2k-1} y_{2k-1})\big)^{-1}\Big)^{-1}$$

in the abelian group $K_\varphi / \Gamma_\varphi$. This coincides with the element

$$f_\varphi(\mathbf{x}) \in K_\varphi / \Gamma_\varphi$$

from (12) proving the Theorem. □

Corollary 2.7. *The universal determinant invariant*

$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

agrees with the composition

$$\det \circ f_{GL(q)} \circ h_2 : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

where the homomorphism

$$f_{GL(q)} : H_2(I_{GL(q)}) \to K_{GL(q)}/\Gamma_{GL(q)}$$

is induced by the central extension (10).

3. The Multiplicative Character

3.1. **Normalized Singular Homology.** This section has three parts. We start by giving a slightly different description of the homology of the simplicial set $GL(A_*)/GL(A)$ studied in [10]. We then explain how the corresponding relative homology groups can be understood using a cokernel complex. At last we expose the connections to group homology. The first part is background material for the relative Chern character and hence for the multiplicative character. The latter two parts are needed for the proof of our main theorem.

Let A be a unital Banach algebra. For each $n \in \{1, 2, \ldots\}$ we let Δ^n denote the standard n-simplex

$$\Delta^n = \{t \in [0,1]^n \mid \sum_{i=1}^{n} t_i \leq 1\}$$

The vertices of Δ^n are given by

$$\mathbf{i} = \begin{cases} (0, \ldots, 0) & \text{for} \quad i = 0 \\ (0, \ldots, 1, \ldots, 0) & \text{for} \quad i \in \{1, \ldots, n\} \end{cases}$$

where the 1 is in position i for $i \in \{1, \ldots, n\}$. Let $C^2(\Delta^n, A)$ be the Banach algebra of C^2 maps $\sigma : \Delta^n \to A$. The norm on $C^2(\Delta^n, A)$ is defined by

$$\|\sigma\| = \|\sigma\|_\infty + \sum_{i=1}^{n} \|D_i(\sigma)\|_\infty + \sum_{1 \leq i \leq j \leq n} \|D_j D_i(\sigma)\|_\infty$$

Here $\| \cdot \|_\infty : C(\Delta^n, A) \to [0, \infty)$ is the supremum norm on the unital Banach algebra of continuous maps from Δ^n to A and

$$D_i : C^2(\Delta^n, A) \to C^1(\Delta^n, A)$$

is the partial differential operator $D_i = \frac{\partial}{\partial t_i}$.

Definition 3.1. *By a C^2 map $\sigma : \Delta^n \to GL(A)$ we will understand an element in the group $GL\big(C^2(\Delta^n, A)\big)$.*

For each $n \in \{1, 2, \ldots\}$ let $GL\big(C^2_0(\Delta^n, A)\big)$ be the normal subgroup of $GL\big(C^2(\Delta^n, A)\big)$ given by

$$GL\big(C^2_0(\Delta^n, A)\big) = \big\{\sigma \in GL\big(C^2(\Delta^n, A)\big) \mid \sigma(\mathbf{0}) = 1\big\}$$

Let $\big(C(A_0(\Delta)), d^N\big)$ be the chain complex with chains

$$C_n\big(A_0(\Delta)\big) = \mathbb{Z}\big[GL\big(C^2_0(\Delta^n, A)\big)\big] \qquad n \in \{1, 2, \ldots\}$$

and with boundary

$$d^{\mathrm{N}} : C_n\big(A_0(\Delta)\big) \to C_{n-1}\big(A_0(\Delta)\big) \qquad d^{\mathrm{N}}(\sigma) = \textstyle\sum_{i=1}^n (-1)^i d_i(\sigma) + d_0(\sigma)\sigma(\mathbf{1})^{-1}$$

where the face operators are given by

$$d_i(\sigma)(t_1, \ldots, t_{n-1}) = \begin{cases} \sigma(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) & \text{for} \quad i \in \{1, \ldots, n\} \\ \sigma(1 - \sum_{j=1}^{n-1} t_j, t_1, \ldots, t_{n-1}) & \text{for} \qquad i = 0 \end{cases}$$

Lemma 3.2.

$$(d^N)^2 = 0$$

Proof. By inspection we get the equalities

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for} \quad i < j$$

for the face operators. See also [13, Chapter 1] for example.

The lemma is therefore proved if we can show that

$$(d_0 \circ d_j)(\sigma) \cdot \big(d_j(\sigma)(\mathbf{1})\big)^{-1} = d^N_{j-1}\big(d_0(\sigma) \cdot \sigma(\mathbf{1})^{-1}\big) \tag{13}$$

for all $j \in \{1, \ldots, n\}$ and $\sigma \in GL\big(A_0(\Delta^n)\big)$. See [21, Lemma 1.0.7].

For $j \in \{2, \ldots, n\}$ we have

$$d_j(\sigma)(\mathbf{1}) = \sigma(\mathbf{1}) \quad \text{and} \quad d^N_{j-1}\big(d_0(\sigma) \cdot \sigma(\mathbf{1})^{-1}\big) = (d_{j-1} \circ d_0)(\sigma) \cdot \sigma(\mathbf{1})^{-1}$$

proving (13) in this case. For $j = 1$ we have

$$d_1(\sigma)(\mathbf{1}) = \sigma(\mathbf{2}) \quad \text{and} \quad d_0\big(d_0(\sigma) \cdot \sigma(\mathbf{1})^{-1}\big) \cdot \big((d_0(\sigma) \cdot \sigma(\mathbf{1})^{-1})(\mathbf{1})\big)^{-1} = (d_0 \circ d_0)(\sigma) \cdot \sigma(\mathbf{2})^{-1}$$

proving (13) in this case as well. $\square$

Definition 3.3. *By* the normalized singular homology of $GL(A)$ *we will understand the homology of the chain complex*

$$\big(C\big(A_0(\Delta)\big), d^N\big)$$

The homology groups are denoted by $H\big(A_0(\Delta)\big)$

Remark 3.4. *The normalized singular homology of $GL(A)$ is isomorphic to the homology of the simplicial set $GL(A_*)/GL(A)$ considered in* [10].

3.1.1. *Relative Normalized Singular Homology.* Let A and B be unital Banach algebras. Suppose that $\varphi : A \to B$ is a unital continuous algebra homomorphism. Let $\sigma : \Delta^n \to A$ be a C^2 map. The composition

$$\varphi \circ \sigma : \Delta^n \to B$$

is then a C^2 map. The partial derivatives are given by the formula

$$\frac{\partial(\varphi \circ \sigma)}{\partial t_i} = \varphi \circ \frac{\partial \sigma}{\partial t_i}$$

for each $i \in \{1, \ldots, n\}$.

It follows that $\varphi : A \to B$ induces a continuous unital algebra homomorphism

$$\varphi(\Delta) : C^2(\Delta^n, A) \to C^2(\Delta^n, B) \qquad \varphi(\Delta)(\sigma) = \varphi \circ \sigma$$

We therefore get a group homomorphism

$$\varphi(\Delta) : GL\big(C_0^2(\Delta^n, A)\big) \to GL\big(C_0^2(\Delta^n, B)\big) \tag{14}$$

which in turn extends by linearity to a chain map

$$\varphi(\Delta)_* : C\big(A_0(\Delta)\big) \to C\big(B_0(\Delta)\big)$$

As in the group homological case we introduce the relative homology groups

$$H_*\big(A_0(\Delta), B_0(\Delta)\big) = H_*\big(\mathrm{Tot}\big(\mathrm{Cone}(\varphi(\Delta)[-1])\big)\big)$$

which fit in the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi(\Delta)_*"] & H_{n+1}\big(B_0(\Delta)\big) \arrow[r, "\partial"] & H_n\big(A_0(\Delta), B_0(\Delta)\big) \arrow[r, "p"] & H_n\big(A_0(\Delta)\big) \arrow[d, "\varphi(\Delta)_*"] \\
\cdots & H_{n-1}\big(A_0(\Delta)\big) \arrow[l, "\varphi(\Delta)_*"'] & H_{n-1}\big(A_0(\Delta), B_0(\Delta)\big) \arrow[l, "p"'] & H_n\big(B_0(\Delta)\big) \arrow[l, "\partial"']
\end{tikzcd}$$

terminating at $H_0\big(A_0(\Delta), B_0(\Delta)\big)$.

For each $n \in \{1, 2, \ldots\}$ let

$$GL\big(C_0^2(\Delta^n, A)\big) \times_{GL\left(C_0^2(\Delta^n, B)\right)} GL\big(C_0^2(\Delta^n, A)\big)$$

denote the set of pairs

$$(\sigma_1, \sigma_2) \in GL\big(C_0^2(\Delta^n, A)\big) \times GL\big(C_0^2(\Delta^n, A)\big)$$

which coincide in $GL\big(C_0^2(\Delta^n, B)\big)$

$$\varphi(\Delta)(\sigma_1) = \varphi(\Delta)(\sigma_2) \in GL\big(C_0^2(\Delta^n, B)\big)$$

Let $\mathrm{Coker}_n\big(\delta_\varphi(\Delta)\big)$ denote the abelian group

$$\mathbb{Z}\big[GL\big(C_0^2(\Delta^n, A)\big) \times_{GL\left(C_0^2(\Delta^n, B)\right)} GL\big(C_0^2(\Delta^n, A)\big)\big]$$

modulo the relation

$$(\sigma_1, \sigma_2) + (\sigma_2, \sigma_3) \sim (\sigma_1, \sigma_3)$$

We get a chain complex $\big(\mathrm{Coker}_n\big(\delta_\varphi(\Delta)\big), d^{\mathrm{N}}\big)$ with boundary

$$d^{\mathrm{N}} : \mathrm{Coker}_n\big(\delta_\varphi(\Delta)\big) \to \mathrm{Coker}_{n-1}\big(\delta_\varphi(\Delta)\big)$$

given on generators as

$$d^{\mathrm{N}}(\sigma_1,\sigma_2) = \sum_{j=1}^{n}(-1)^j\big(d_j(\sigma_1), d_j(\sigma_2)\big) + \big(d_0(\sigma_1)\cdot\sigma_1(\mathbf{1})^{-1}, d_0(\sigma_2)\cdot\sigma_2(\mathbf{1})^{-1}\big)$$

In analogy with the group homological case we have the theorem

Theorem 3.5. *Suppose that the induced map*

$$\varphi(\Delta) : GL\big(C_0^2(\Delta^n, A)\big) \to GL\big(C_0^2(\Delta^n, B)\big)$$

is surjective for each $n \in \{1,2,\ldots\}$. Then the map

$$i\circ\varepsilon : \mathrm{Coker}_*\big(\delta_\varphi(\Delta)\big) \to \mathrm{Tot}_*\big(\mathrm{Cone}\big(\varphi(\Delta)\big)[-1]\big)$$

which on generators is given by

$$(i\circ\varepsilon)(\sigma_1,\sigma_2) = (0, \sigma_1 - \sigma_2)$$

induces an isomorphism in homology.

The next lemma shows that the condition in Theorem 3.5 is satisfied in a variety of situations

Lemma 3.6. *Suppose that the continuous unital algebra homomorphism $\varphi : A \to B$ has a continuous linear section*

$$s : B \to A \qquad \varphi\circ s = 1_B$$

then the induced map

$$\varphi(\Delta) : GL\big(C_0^2(\Delta^n, A)\big) \to GL\big(C_0^2(\Delta^n, B)\big)$$

is surjective for all $n \in \{1,2,\ldots\}$

Proof. For each $n \in \{1,2,\ldots\}$ we can find bijections ψ_A and ψ_B such that the diagram

$$\begin{tikzcd}
GL\big(C_0^2(\Delta^n, A)\big) \arrow[r, "\psi_A"] \arrow[d, "\varphi(\Delta)"'] & GL\big(C_0^2(\Delta^1, C^2(\Delta^{n-1}, A))\big) \arrow[d, "\varphi(\Delta)"] \\
GL\big(C_0^2(\Delta^n, B)\big) \arrow[r, "\psi_B"'] & GL\big(C_0^2(\Delta^1, C^2(\Delta^{n-1}, B))\big)
\end{tikzcd}$$

commutes. This reduces the problem to finding lifts of C^2 maps

$$\sigma : \Delta^1 \to GL\big(C^2(\Delta^{n-1}, B)\big) \qquad \sigma(\mathbf{0}) = 1$$

The continuous linear section induces a continuous linear map

$$s(\Delta^{n-1}) : C^2(\Delta^{n-1}, B) \to C^2(\Delta^{n-1}, A)$$

such that $\varphi(\Delta^{n-1})\circ s(\Delta^{n-1}) = 1_{C^2(\Delta^{n-1},B)}$. This gives us the surjectivity of the unital Banach algebra homomorphism

$$\varphi(\Delta) : C^2(\Delta^{n-1}, A) \to C^2(\Delta^{n-1}, B)$$

The lemma now follows by a slight modification of standard results on liftings. See [2, Corollary 3.4.4] for example. □

3.1.2. *Relating Normalized Singular Homology and Group Homology.* Let A be a unital Banach algebra. Let $GL_0(A)$ denote the connected component of the identity in $GL(A)$. Thus each element in $GL_0(A)$ has a representative $T \in GL_m(A)$ such that T is connected to $1 \in GL_m(A)$ through a continuous path of invertibles.

Lemma 3.7. *The assignment*

$$\sigma \mapsto \big(\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \ldots, \sigma(\mathbf{n-1})\sigma(\mathbf{n})^{-1}\big)$$

defines a chain map

$$\theta : C_*\big(A_0(\Delta)\big) \to C_*\big(GL_0(A)\big)$$

Here $\mathbf{0}, \ldots, \mathbf{n} \in \Delta^n$ denote the vertices of the standard n-simplex.

Proof. This is immediate by inspection. In fact we check for each $\sigma \in GL\big(C_0^2(\Delta^n, A)\big)$ that

$$(d_i \circ \theta)(\sigma) = \begin{cases} (\theta \circ d_i)(\sigma) & \text{for} \quad i \in \{1, \ldots, n\} \\ (\theta \circ d_0)\big(\sigma \cdot \sigma(\mathbf{1})^{-1}\big) & \text{for} \qquad i = 0 \end{cases}$$

See also [18, Paragraphe 6.15-6.18]. Note that $\theta(\sigma) \in GL_0(A)^n$ is immediate since $\sigma(\mathbf{0}) = 1$ for each $\sigma \in GL\big(C_0^2(\Delta^n, A)\big)$. □

Suppose that $\varphi : A \to B$ is a continuous unital algebra homomorphism between the unital Banach algebras A and B. We can then define a relative version of the chain map θ

$$\theta : H_n\big(A_0(\Delta), B_0(\Delta)\big) \to H_n\big(GL_0(A), GL_0(B)\big)$$

by the assignment

$$\theta : (\sigma, \tau) \mapsto \big(\theta(\sigma), \theta(\tau)\big) \qquad (\tau, \sigma) \in C_{n+1}\big(B_0(\Delta)\big) \oplus C_n\big(A_0(\Delta)\big)$$

Likewise we have homomorphisms between the homology theories of the cokernel complexes

$$\theta : H_{n-1}\big(\mathrm{Coker}(\delta_\varphi(\Delta))\big) \to H_{n-1}\big(\mathrm{Coker}(\delta_\varphi)\big)$$

given on generators as

$$\theta(\sigma_1, \sigma_2) = \big(\theta(\sigma_1), \theta(\sigma_2))\big)$$

The relations between these homomorphisms are well expressed in the commutative diagram

$$\begin{tikzcd} H_n\big(B_0(\Delta)\big) \arrow[r, "\partial"] \arrow[d, "\theta"'] & H_{n-1}\big(A_0(\Delta), B_0(\Delta)\big) \arrow[d, "\theta"'] & H_{n-1}\big(\mathrm{Coker}(\delta_\varphi(\Delta))\big) \arrow[l, "i\circ\varepsilon"'] \arrow[d, "\theta"] \\ H_n\big(GL_0(B)\big) \arrow[r, "\partial"'] & H_{n-1}\big(GL_0(A), GL_0(B)\big) & H_{n-1}\big(\mathrm{Coker}(\delta_\varphi)\big) \arrow[l, "i\circ\varepsilon"] \end{tikzcd} \tag{15}$$

In the case where $\varphi : A \to B$ is surjective it follows by [2, Corollary 3.4.4] that the connected component of the identity, $GL_0(B)$, is a normal subgroup of the image, $I_{GL(\varphi)}$, of the induced map $GL(\varphi) : GL(A) \to GL(B)$. This observation yields homomorphisms

$$H_n\big(GL_0(B)\big) \to H_n(I_{GL(\varphi)}) \quad \text{and} \quad H_{n-1}\big(GL_0(A), GL_0(B)\big) \to H_{n-1}(GL(A), I_{GL(\varphi)})$$

which we will normally suppress.

For details on the chain maps and homology theories involved we refer to Section 2.1 and Section 3.1.1.

3.2. **Continuous Cyclic Homology and The Relative Chern Character.** In this section, we will review the definitions of continuous cyclic homology and the relative Chern character. For details on cyclic homology the reader should consult the references [7, 8, 18, 21]. Details on the relative Chern character can be found in [10, 18, 30].

Let A be a unital Banach algebra. Let $C_n^{\text{cont}}(A)$ be the $\mathbb{C}$-module

$$C_n^{\text{cont}}(A) = \underbrace{A \hat{\otimes} \ldots \hat{\otimes} A}_{n+1} \qquad n \in \{0, 1, 2, \ldots\}$$

Here $\hat{\otimes}$ denotes the projective tensor product in the sense of Grothendieck [17]. The cyclic operator

$$t : C_n^{\text{cont}}(A) \to C_n^{\text{cont}}(A)$$

is defined on simple tensors by

$$t : a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

Let $C_n^{\lambda,\text{cont}}(A)$ denote the quotient $\mathbb{C}$-module

$$C_n^{\lambda,\text{cont}}(A) = C_n^{\text{cont}}(A)/(1-t)$$

The continuous cyclic homology of the Banach algebra A is the homology of the chain complex $\big(C_n^{\lambda,\text{cont}}(A), b\big)$ with boundary

$$b : C_n^{\lambda,\text{cont}}(A) \to C_{n-1}^{\lambda,\text{cont}}(A)$$

given on simple tensors by

$$\begin{aligned} &b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) \\ &\quad = (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \end{aligned}$$

We denote the homology groups by $HC_n^{\text{cont}}(A)$.

Each continuous algebra homomorphism $\varphi : A \to B$ induces a group homomorphism $\varphi_* : HC_*^{\text{cont}}(A) \to HC_*^{\text{cont}}(B)$ which on chains is extended from

$$\varphi_* : a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto \varphi(a_0) \otimes \varphi(a_1) \otimes \ldots \otimes \varphi(a_n)$$

As in the group homological case we introduce the relative homology groups

$$HC_*^{\text{cont}}(A, B) = H_*\big(\text{Tot}(\text{Cone}(\varphi)[-1])\big)$$

which fit in the long exact sequence

$$
(16)\qquad
\begin{array}{ccccccc}
\cdots & \xrightarrow{\varphi_*} & HC_{n+1}^{\mathrm{cont}}(B) & \xrightarrow{\partial} & HC_n^{\mathrm{cont}}(A,B) & \xrightarrow{p} & HC_n^{\mathrm{cont}}(A) \\
 & & & & & & \downarrow{\scriptstyle \varphi_*} \\
\cdots & \xleftarrow[\varphi_*]{} & HC_{n-1}^{\mathrm{cont}}(A) & \xleftarrow[p]{} & HC_{n-1}^{\mathrm{cont}}(A,B) & \xleftarrow[\partial]{} & HC_n^{\mathrm{cont}}(B)
\end{array}
$$

Let $\mathrm{Ker}_n(\varphi)$ denote the $\mathbb{C}$-module

$$\mathrm{Ker}_n(\varphi) = \mathrm{Ker}\big(\varphi : C_n^{\lambda,\mathrm{cont}}(A) \to C_n^{\lambda,\mathrm{cont}}(B)\big) \qquad n \in \{0,1,2,\ldots\}$$

We get a chain complex

$$\big(\mathrm{Ker}_n(\varphi), b\big)$$

with boundary induced by the boundary on $C^{\lambda,\mathrm{cont}}(A)$. The homology groups of the kernel complex are denoted by $HC\big(\mathrm{Ker}(\varphi)\big)$.

Lemma 3.8. *Suppose that $\varphi : A \to B$ has a continuous linear section*

$$s : B \to A \qquad \varphi \circ s = 1_B$$

then the map

$$i : \mathrm{Ker}_*(\varphi) \to \mathrm{Tot}_*\big(\mathrm{Cone}(\varphi)[-1]\big) \qquad i(x) = (0,x)$$

induces an isomorphism on homology.

Proof. The existence of the section entails the surjectivity of the chain map

$$\varphi_* : C_*^{\lambda,\mathrm{cont}}(A) \to C_*^{\lambda,\mathrm{cont}}(B)$$

in all degrees. The lemma now follows by standard homological arguments, see [21, Proposition 1.0.12] for example. □

3.2.1. *The Relative Chern Character.* For a unital Banach algebra A, the topological K-theory $K_*^{\mathrm{top}}(A)$ and the algebraic K-theory $K_*(A)$ can be compared by introducing the relative K-theory $K_*^{\mathrm{rel}}(A)$. The three K-theories fit together in a long exact sequence

$$
(17)\qquad
\begin{array}{ccccccc}
\cdots & \xrightarrow{i} & K_{n+1}^{\mathrm{top}}(A) & \xrightarrow{v} & K_n^{\mathrm{rel}}(A) & \xrightarrow{\theta} & K_n(A) \\
 & & & & & & \downarrow{\scriptstyle i} \\
\cdots & \xleftarrow[i]{} & K_{n-1}(A) & \xleftarrow[\theta]{} & K_{n-1}^{\mathrm{rel}}(A) & \xleftarrow[v]{} & K_n^{\mathrm{top}}(A)
\end{array}
$$

which terminates at $K_1^{\mathrm{top}}(A)$.

This long exact sequence is related to the continuous SBI-sequence in homology by means of Chern characters

$$
\begin{array}{ccccccccc}
K_{n+1}^{\mathrm{top}}(A) & \xrightarrow{v} & K_n^{\mathrm{rel}}(A) & \xrightarrow{\theta} & K_n(A) & \xrightarrow{i} & K_n^{\mathrm{top}}(A) & \xrightarrow{v} & K_{n-1}^{\mathrm{rel}}(A) \\
{\scriptstyle \mathrm{ch}_{n+1}^{\mathrm{top}}}\downarrow & & {\scriptstyle \mathrm{ch}_n^{\mathrm{rel}}}\downarrow & & {\scriptstyle \frac{1}{n!}D_n}\downarrow & & {\scriptstyle \mathrm{ch}_n^{\mathrm{top}}}\downarrow & & {\scriptstyle \mathrm{ch}_{n-1}^{\mathrm{rel}}}\downarrow \\
HC_{n+1}^{\mathrm{cont}}(A) & \xrightarrow[S]{} & HC_{n-1}^{\mathrm{cont}}(A) & \xrightarrow[\frac{1}{n}B]{} & HH_n^{\mathrm{cont}}(A) & \xrightarrow[I]{} & HC_n^{\mathrm{cont}}(A) & \xrightarrow[S]{} & HC_{n-2}^{\mathrm{cont}}(A)
\end{array}
$$

In this section we recall the construction of the relative Chern character

$$\mathrm{ch}_n^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to HC_{n-1}^{\mathrm{cont}}(A)$$

The relative Chern character is the composition of three homomorphisms.

The first one is the Hurewicz homomorphism

$$h_n^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to H_n\big(A_0(\Delta)\big)$$

Here $H_n\big(A_0(\Delta)\big)$ is the normalized singular homology group introduced in section 3.1. Details on the Hurewicz homomorphism can be found in [14] or [32] among others.

The second one is the logarithm

$$\gamma : H_n\big(A_0(\Delta)\big) \to \lim_{m\to\infty} HC_{n-1}^{\mathrm{cont}}\big(M_m(A)\big)$$

which is induced by

$$\gamma : \sigma \mapsto \frac{(-1)^n}{n!} \sum_{s\in\Sigma_n} \mathrm{sgn}(s) \int_{\Delta^n} \frac{\partial\sigma}{\partial t_{s(1)}} \cdot \sigma^{-1} \otimes \ldots \otimes \frac{\partial\sigma}{\partial t_{s(n)}} \cdot \sigma^{-1} dt_1 \wedge \ldots \wedge dt_n \tag{18}$$

for each C^2 map $\sigma : \Delta^n \to GL_m(A)$. Note that the direct limit is taken over the maps

$$HC_{n-1}^{\mathrm{cont}}\big(M_m(A)\big) \to HC_{n-1}^{\mathrm{cont}}\big(M_{m+1}(A)\big)$$

induced by the inclusion $x \mapsto x \oplus 0$.

The last one is the generalized trace on continuous cyclic homology

$$\mathrm{TR} : \lim_{m\to\infty} HC^{\mathrm{cont}}\big(M_m(A)\big) \to HC^{\mathrm{cont}}(A)$$

See for instance [20].

The relative Chern character

$$\mathrm{ch}_n^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \to HC_{n-1}^{\mathrm{cont}}(A)$$

is thus defined as the composition

$$\mathrm{ch}_n^{\mathrm{rel}} = \mathrm{TR} \circ \gamma \circ h_n^{\mathrm{rel}} \tag{19}$$

of the Hurewicz homomorphism, the logarithm and the generalized trace.

Remark 3.9. *In the papers* [10] *and* [30] *it is emphasized that the map*

$$\gamma : H_*\big(A_0(\Delta)\big) \to HC_{*-1}^{\mathrm{cont}}(A)$$

which we refer to as the logarithm, actually factorizes through either a completed version of the non-commutative De Rham homology or continuous Lie algebra homology. We have chosen to simplify as much as possible thus ignoring these important factorization results.

3.3. **The Universal Multiplicative Character.** In this section we recall the construction of the odd universal multiplicative character. The main ingredients are the relative Chern character of Section 3.2.1, a calculation of a topological K-group together with the long exact sequence (17) and the cyclic cocycle of a universal Fredholm module. We refer to [10] for details.

Let H be a separable Hilbert space. For each $p \in \{1, 2, \ldots\}$ let $\mathcal{M}^{2p-1}$ denote the $\mathbb{C}$-subalgebra of $\mathcal{L}(H \oplus H)$ consisting of operators of the form

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathcal{L}(H \oplus H)$$

with $x_{12}, x_{21} \in \mathcal{L}^{2p}(H)$ in the $2p^{\text{th}}$ Schatten ideal. The $\mathbb{C}$-algebra $\mathcal{M}^{2p-1}$ becomes a Banach algebra when equipped with the norm

$$\|x\| = \|x\|_\infty + \| \, [F, x] \, \|_{2p}$$

where F is the operator

$$F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the norms $\| \cdot \|_\infty$ and $\| \cdot \|_{2p}$ are the operator norm and the norm on the $2p^{\text{th}}$ Schatten ideal respectively. For details on the Schatten ideals we refer to [29].

The continuous linear map

$$\tau_{2p-1} : \mathcal{M}^{2p-1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{M}^{2p-1} \to \mathbb{C}$$

defined on simple tensors by

$$\begin{aligned} &\tau_{2p-1}(x^0 \otimes \ldots \otimes x^{2p-1}) \\ &\quad = (-1)^{p-1} \frac{(2p-1)!}{(p-1)!} \mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & x^0_{12} \\ x^0_{21} & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 0 & x^{2p-1}_{12} \\ x^{2p-1}_{21} & 0 \end{pmatrix} \end{aligned}$$

determines a continuous cyclic cocycle and thereby a homomorphism

$$\tau_{2p-1} : HC^{\text{cont}}_{2p-1}(\mathcal{M}^{2p-1}) \to \mathbb{C}$$

Here $\mathrm{Tr} : M_2\big(\mathcal{L}^1(H)\big) \to \mathbb{C}$ is the operator algebraic trace. For details see [9] and [10].

Pre-composition with the relative Chern character thus yields a homomorphism

$$\tau_{2p-1} \circ \mathrm{ch}^{\text{rel}}_{2p} : K^{\text{rel}}_{2p}(\mathcal{M}^{2p-1}) \to \mathbb{C}$$

In [10] it is shown that $K_{2p}(\mathcal{M}^{2p-1})$ is the cokernel of

$$v : K^{\text{top}}_{2p+1}(\mathcal{M}^{2p-1}) \to K^{\text{rel}}_{2p}(\mathcal{M}^{2p-1})$$

from the exact sequence (17). Furthermore it is shown that, when quotienting out by the additive subgroup $(2\pi i)^p \mathbb{Z}$ of $\mathbb{C}$, the character

$$\tau_{2p-1} \circ \mathrm{ch}^{\text{rel}}_{2p} : K^{\text{rel}}_{2p}(\mathcal{M}^{2p-1}) \to \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

descends to a homomorphism

$$\tau_{2p-1} \circ \mathrm{ch}^{\text{rel}}_{2p} : K_{2p}(\mathcal{M}^{2p-1}) \to \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

This is the odd universal multiplicative character

$$M_U : K_{2p}(\mathcal{M}^{2p-1}) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

4. Comparing The Determinant Invariant and The Multiplicative Character

4.1. **Factorization through Relative Cyclic Homology.** In this section we show that the composition

$$\tau_1 \circ \mathrm{ch}_2^{\mathrm{rel}} : K_2^{\mathrm{rel}}(\mathcal{M}^1) \to \mathbb{C}$$

factorizes through relative continuous cyclic homology. The results can be found in condensed form in [10, Section 5].

Let H be a separable Hilbert space and let $\mathcal{M}^1$ be the Banach algebra considered in Section 3.3. Let P be the projection

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H)$$

and let $q : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{L}^1(H)$ be the quotient map. The map

$$R : \mathcal{M}^1 \to \mathcal{L}(H)/\mathcal{L}^1(H)$$

given by

$$R : x \mapsto q(PxP) \quad \text{for all} \quad x \in \mathcal{M}^1$$

is then an algebra homomorphism.

Let $\mathcal{T}^1 \subseteq \mathcal{L}(H) \times \mathcal{M}^1$ be the $\mathbb{C}$-subalgebra such that $(S, x) \in \mathcal{T}^1$ precisely when $S - PxP \in \mathcal{L}^1(H)$. The diagram

$$\begin{tikzcd} \mathcal{T}^1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{M}^1 \arrow[d, "\alpha"] \\ \mathcal{L}(H) \arrow[r, "q"'] & \mathcal{L}(H)/\mathcal{L}^1(H) \end{tikzcd} \tag{20}$$

is thus a commutative diagram of $\mathbb{C}$-algebras. Here π_1 and π_2 are the projections given by $\pi_1(S, x) = S$ and $\pi_2(S, x) = x$.

The $\mathbb{C}$-algebra $\mathcal{T}^1$ becomes a Banach algebra when equipped with the norm

$$\|(S, x)\| = \|PxP - S\|_1 + \|x\|_\infty + \|[F, x]\|_2$$

The kernel of the surjective continuous algebra homomorphism $\pi_2 : \mathcal{T}^1 \to \mathcal{M}^1$ is the operators of trace class $\mathcal{L}^1(H)$. Therefore the sequence of Banach algebras

$$0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{\ i\ } \mathcal{T}^1 \xrightarrow{\ \pi_2\ } \mathcal{M}^1 \longrightarrow 0 \tag{21}$$

is exact.

The continuous algebra homomorphism π_2 has a continuous linear section

$$s : \mathcal{M}^1 \to \mathcal{T}^1 \qquad x \mapsto (x_{11}, x)$$

It follows by Lemma 3.8 that the continuous relative cyclic homology of π_2 is isomorphic to the continuous cyclic homology of the kernel

$$HC_*^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1) \cong HC_*^{\mathrm{cont}}\big(\mathrm{Ker}(\pi_2)\big)$$

In particular each element in $HC_0^{\text{cont}}(\mathcal{T}^1, \mathcal{M}^1)$ can be represented by an operator of trace class

$$(S,0) \in \mathcal{T}^1 \qquad S \in \mathcal{L}^1(H)$$

Lemma 4.1. *The continuous linear map defined by*

$$T : (S,0) \mapsto \mathrm{Tr}(S) \quad \textit{for all} \quad (S,0) \in \mathcal{T}^1$$

induces a map on continuous relative cyclic homology

$$T : HC_0^{\text{cont}}(\mathcal{T}^1, \mathcal{M}^1) \to \mathbb{C}$$

Here $\mathrm{Tr} : \mathcal{L}^1(H) \to \mathbb{C}$ *is the operator algebraic trace.*

Proof. Suppose that $[w] \in C_1^{\lambda,\text{cont}}(\mathcal{T}^1)$ maps to zero under

$$(\pi_2)_* : C_1^{\lambda,\text{cont}}(\mathcal{T}^1) \to C_1^{\lambda,\text{cont}}(\mathcal{M}^1)$$

Let $S \in \mathcal{L}^1(H)$ be such that

$$b[w] = (S,0) \in \mathcal{T}^1$$

Pick a representative $w \in \mathcal{T}^1 \hat{\otimes} \mathcal{T}^1$ of $[w]$. Let

$$\sum_{i=1}^{n_k} (T_i^k + (x_i^k)_{11}, x_i^k) \otimes (S_i^k + (y_i^k)_{11}, y_i^k) \in \mathcal{T}^1 \otimes \mathcal{T}^1$$

be a sequence converging to $w \in \mathcal{T}^1 \hat{\otimes} \mathcal{T}^1$ for $k \to \infty$. By continuity it follows that

$$\sum_{i=1}^{n_k} \big([T_i^k, S_i^k] + [(x_i^k)_{11}, S_i^k] + [T_i^k, (y_i^k)_{11}] + (y_i^k)_{12}(x_i^k)_{21} - (x_i^k)_{12}(y_i^k)_{21} \big)$$

converges to S in $\mathcal{L}^1(H)$. Since the operator algebraic trace $\mathrm{Tr} : \mathcal{L}^1(H) \to \mathbb{C}$ is continuous we have

$$\begin{aligned} &\sum_{i=1}^{n_k} \mathrm{Tr}\big([T_i^k, S_i^k] + [(x_i^k)_{11}, S_i^k] + [T_i^k, (y_i^k)_{11}] + (y_i^k)_{12}(x_i^k)_{21} - (x_i^k)_{12}(y_i^k)_{21} \big) - \mathrm{Tr}(S) \\ &\quad = \sum_{i=1}^{n_k} \mathrm{Tr}\big((y_i^k)_{12}(x_i^k)_{21} - (x_i^k)_{12}(y_i^k)_{21} \big) - \mathrm{Tr}(S) \to 0 \end{aligned}$$

But the trace of the sum

$$\mathrm{Tr}\Big(\sum_{i=1}^{n_k} (y_i^k)_{12}(x_i^k)_{21} - (x_i^k)_{12}(y_i^k)_{21} \Big)$$

is precisely the image of

$$- \sum_{i=1}^{n_k} (T_i^k + (x_i^k)_{11}, x_i^k) \otimes (S_i^k + (y_i^k)_{11}, y_i^k) \in \mathcal{T}^1 \otimes \mathcal{T}^1$$

under the map $\tau_1 \circ \pi_2 : C_1^{\lambda,\text{cont}}(\mathcal{T}^1) \to \mathbb{C}$. By continuity it follows that

$$\mathrm{Tr}\Big(\sum_{i=1}^{n_k} (y_i^k)_{12}(x_i^k)_{21} - (x_i^k)_{12}(y_i^k)_{21} \Big) \to 0$$

and therefore by the uniqueness of the limit $\mathrm{Tr}(S) = 0$ proving the lemma. $\square$

We are now ready to prove that the character

$$\tau_1 : HC_1^{\mathrm{cont}}(\mathcal{M}^1) \to \mathbb{C}$$

factorizes through the relative continuous cyclic homology group $HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1)$

Lemma 4.2. *The diagram*

$$\begin{tikzcd} HC_1^{\mathrm{cont}}(\mathcal{M}^1) \arrow[r, "\partial"] \arrow[d, "\tau_1"'] & HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1) \arrow[dl, "T"] \\ \mathbb{C} & \end{tikzcd}$$

is commutative. Or in equations

$$\tau_1 = T \circ \partial$$

Proof. Let $[w] \in HC_1^{\mathrm{cont}}(\mathcal{M}^1)$ be represented by the element

$$w \in \mathcal{M}^1 \hat{\otimes} \mathcal{M}^1$$

Choose a sequence

$$w^k = \sum_{i=1}^{n_k} x_i^k \otimes y_i^k \in \mathcal{M}^1 \otimes \mathcal{M}^1$$

converging to w. By the continuity of the character

$$\tau_1 : \mathcal{M}^1 \hat{\otimes} \mathcal{M}^1 \to \mathbb{C}$$

we have the convergence

$$\tau_1(w^k) = \sum_{i=1}^{n_k} \mathrm{Tr}\big((x_i^k)_{12}(y_i^k)_{21} - (x_i^k)_{21}(y_i^k)_{12}\big) \to \tau_1[w] \tag{22}$$

Since $b(w) = 0$ and since the linear section of π_2

$$s : \mathcal{M}^1 \to \mathcal{T}^1 \qquad x \mapsto (x_{11}, x)$$

is continuous we have the convergence

$$-b\Big(\sum_{i=1}^{n_k} s(x_i^k) \otimes s(y_i^k)\Big) = -b\Big(\sum_{i=1}^{n_k} \big((x_i^k)_{11}, x_i^k\big) \otimes \big((y_i)_{11}^k, y_i^k\big)\Big) \to (S, 0) \tag{23}$$

for some $(S, 0) \in \mathcal{T}^1$. The element $(S, 0)$ must represent the boundary of $[w]$ so we deduce the equality

$$\mathrm{Tr}(S) = (T \circ \partial)[w] \tag{24}$$

Now the convergence of (23) is equivalent to the convergence

$$\sum_{i=1}^{n_k} \big((x_i^k)_{12}(y_i^k)_{21} - (y_i^k)_{12}(x_i^k)_{21}\big) \to S$$

in $\mathcal{L}^1(H)$. By continuity of the operator algebraic trace we therefore have

$$\sum_{i=1}^{n_k} \mathrm{Tr}\big((x_i^k)_{12}(y_i^k)_{21} - (y_i^k)_{12}(x_i^k)_{21}\big) \to \mathrm{Tr}(S)$$

So by (22) and (24)

$$(T \circ \partial)[w] = \mathrm{Tr}(S) = \tau_1[w]$$

as desired. □

To draw some immediate consequences of Lemma 4.2 we define a relative logarithm

$$\gamma : H_n\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \to \lim_{m\to\infty} HC_{n-1}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big)$$

and a relative generalized trace

$$\mathrm{TR} : \lim_{m\to\infty} HC^{\mathrm{cont}}_{n-1}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big) \to HC_{n-1}(\mathcal{T}^1, \mathcal{M}^1)$$

For each element

$$(\sigma, \tau) \in C_{n+1}\big(\mathcal{M}_0^1(\Delta)\big) \oplus C_n\big(\mathcal{T}_0^1(\Delta)\big)$$

we let

$$\gamma(\sigma,\tau) = \big(\gamma(\sigma), \gamma(\tau)\big) \tag{25}$$

See Section 3.2 (18) for the definition of γ in the non-relative setting. Likewise for each element

$$(x, y) \in C_n^{\lambda,\mathrm{cont}}\big(M_m(\mathcal{M}^1)\big) \oplus C_{n-1}^{\lambda,\mathrm{cont}}\big(M_m(\mathcal{T}^1)\big)$$

we let

$$\mathrm{TR}(x,y) = \big(\mathrm{TR}(x), \mathrm{TR}(y)\big) \tag{26}$$

See also [21, Chapter 1].

Note that by [30, Proposition 3.2] we have

$$(b \circ \gamma)(\sigma) = \tfrac{1}{n-1}(\gamma \circ d^N)(\sigma) \quad \text{for each} \quad \sigma \in GL\big(C_0^2(\Delta^n, A)\big)$$

With these definitions the diagram

$$\begin{tikzcd}
H_{n+1}\big(\mathcal{M}_0^1(\Delta)\big) \arrow[r, "\partial"] \arrow[d, "\mathrm{TR}\circ\gamma"'] & H_n\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \arrow[d, "\mathrm{TR}\circ\gamma"] \\
HC_n^{\mathrm{cont}}(\mathcal{M}^1) \arrow[r, "\partial"'] & HC_{n-1}^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1)
\end{tikzcd} \tag{27}$$

becomes commutative.

Corollary 4.3. *The diagram*

$$\begin{tikzcd}
H_2\big(\mathcal{M}_0^1(\Delta)\big) \arrow[r, "\partial"] \arrow[d, "\tau_1\circ\mathrm{TR}\circ\gamma"'] & H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \arrow[dl, "T\circ\mathrm{TR}\circ\gamma"] \\
\mathbb{C} &
\end{tikzcd}$$

is commutative. In particular we have

$$\tau_1 \circ \mathrm{ch}_2^{\mathrm{rel}} = T \circ \mathrm{TR} \circ \gamma \circ \partial \circ h_2^{\mathrm{rel}}$$

Proof. An immediate consequence of the commutative diagram (27), Lemma 4.2 and the definition of the relative Chern character given in (19). $\square$

4.2. **Determinants and The Relative Logarithm.** In this important part of the paper we start relating the universal multiplicative character with the Fredholm determinant. The main result of this section is thus Lemma 4.7 where we show that the composition of the relative logarithm

$$\gamma : H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \to \lim_{m\to\infty} HC_0^{\mathrm{cont}}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big)$$

with the trace map

$$T \circ \mathrm{TR} : \lim_{m\to\infty} HC_0^{\mathrm{cont}}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big) \to \mathbb{C}$$

and the exponential

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

is a Fredholm determinant.

Definition 4.4. *Let $\mathcal{G}$ denote the kernel of the map*

$$GL(q) : GL\big(\mathcal{L}(H)\big) \to GL\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

induced by the quotient map

$$q : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{L}^1(H)$$

The operators in $\mathcal{G}$ are said to be of determinant class.

Definition 4.5. *By a C^2 map $\sigma : \Delta^n \to \mathcal{G}$ we will understand an element in the group*

$$\sigma \in GL\big(C(\Delta^n, \mathcal{L}(H))\big)$$

such that

$$\sigma - 1 \in M_\infty\big(C^2(\Delta^n, \mathcal{L}^1(H))\big)$$

Since the continuous unital surjective algebra homomorphism $\pi_2 : \mathcal{T}^1 \to \mathcal{M}^1$ has a continuous linear section, it follows by Lemma 3.6 and Theorem 3.5 that

$$i \circ \varepsilon : H_n\big(\mathrm{Coker}(\delta_{\pi_2}(\Delta))\big) \to H_n\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \tag{28}$$

is an isomorphism.

Define the map

$$\tilde{\gamma} : \mathbb{Z}\big[GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big) \times_{GL\left(C_0^2(\Delta^1, \mathcal{M}^1)\right)} GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big)\big] \to \mathbb{C}$$

by the assignment

$$\tilde{\gamma} : (\sigma_1, \sigma_2) \mapsto -\mathrm{Tr}_\infty \int_0^1 \frac{d(\sigma_1\sigma_2^{-1})}{dt} \cdot \sigma_2\sigma_1^{-1} dt$$

for each generator

$$(\sigma_1, \sigma_2) \in GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big) \times_{GL\left(C_0^2(\Delta^1, \mathcal{M}^1)\right)} GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big)$$

Here the trace

$$\mathrm{Tr}_\infty : M_\infty\big(\mathcal{L}^1(H)\big) \to \mathbb{C}$$

is induced by

$$\mathrm{Tr}_m : x \mapsto \textstyle\sum_{i=1}^m \mathrm{Tr}(x_{ii}) \quad \text{for all} \quad x \in M_m\big(\mathcal{L}^1(H)\big)$$

Note that $\sigma_1 \cdot \sigma_2^{-1} : \Delta^1 \to \mathcal{G}$ is C^2 in the sense of Definition 4.5.

Lemma 4.6. *We have the equality*

$$\tilde{\gamma}(\sigma_1, \sigma_2) = \mathrm{Tr}_\infty\big(\int_0^1 \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1} dt - \int_0^1 \frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} dt \big)$$

for all generators

$$(\sigma_1, \sigma_2) \in GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big) \times_{GL\big(C_0^2(\Delta^1, \mathcal{M}^1)\big)} GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big)$$

In particular $\tilde{\gamma}$ induces a map on homology making the diagram

$$\begin{tikzcd} H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \arrow[r, "(i\circ\varepsilon)^{-1}"] \arrow[d, "T\circ\mathrm{TR}\circ\gamma"'] & H_1\Big(\mathrm{Coker}\big(\delta_{\pi_2}(\Delta)\big)\Big) \arrow[dl, "\tilde{\gamma}"] \\ \mathbb{C} & \end{tikzcd} \tag{29}$$

commutative.

Proof. Since the operator algebraic trace

$$\mathrm{Tr}_m : M_m\big(\mathcal{L}^1(H)\big) \to \mathbb{C}$$

is continuous and linear we have

$$-\mathrm{Tr}_\infty \int_0^1 \frac{d(\sigma_1 \sigma_2^{-1})}{dt} \cdot \sigma_2 \sigma_1^{-1} dt = -\int_0^1 \mathrm{Tr}_\infty\big(\frac{d(\sigma_1 \sigma_2^{-1})}{dt} \cdot \sigma_2 \sigma_1^{-1}\big) dt$$

Using the Leibnitz rule for the differential operator $\frac{d}{dt}$ we get

$$\frac{d(\sigma_1 \sigma_2^{-1})}{dt} = \frac{d\sigma_1}{dt} \cdot \sigma_2^{-1} - \sigma_1 \sigma_2^{-1} \cdot \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1}$$

Defining $\alpha \in M_\infty\big(C^2(\Delta^1, \mathcal{L}^1(H))\big)$ by $\alpha + 1 = \sigma_2 \sigma_1^{-1}$ we have

$$\begin{aligned} &\mathrm{Tr}_\infty\big(\frac{d(\sigma_1 \cdot \sigma_2^{-1})}{dt} \cdot \sigma_2 \sigma_1^{-1}\big) \\ &\quad = \mathrm{Tr}_\infty\big(\frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} - (\alpha+1)^{-1} \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1}(\alpha+1)\big) \\ &\quad = \mathrm{Tr}_\infty\big(\frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} - (\alpha+1)^{-1} \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1}\big) - \mathrm{Tr}_\infty\big((\alpha+1)^{-1} \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1} \alpha\big) \\ &\quad = \mathrm{Tr}_\infty\big(\frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} - \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1}\big) \end{aligned}$$

The equality

$$\tilde{\gamma}\big((\sigma_1, \sigma_2)\big) = \mathrm{Tr}_\infty\big(\int_0^1 \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1} dt - \int_0^1 \frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} dt\big)$$

has thus been proven.

To prove the commutativity of the diagram (29) recall that by Theorem 3.5 the isomorphism

$$i \circ \varepsilon : H_n\big(\mathrm{Coker}(\delta_{\pi_2}(\Delta))\big) \to H_n\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big)$$

is induced by

$$(\sigma_1, \sigma_2) \mapsto (0, \sigma_1 - \sigma_2) \in \mathbb{Z}\big[GL\big(C_0^2(\Delta^{n+1}, \mathcal{M}^1)\big)\big] \oplus \mathbb{Z}\big[GL\big(C_0^2(\Delta^n, \mathcal{T}^1)\big)\big]$$

Applying the relative logarithm

$$\gamma : H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \to \lim_{m \to \infty} HC_0^{\mathrm{cont}}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big)$$

defined in Section 4.1 (25) we get

$$\gamma(0, \sigma_1 - \sigma_2) = (0, \int_0^1 \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1} dt - \int_0^1 \frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} dt)$$

Now, the composition $T \circ \mathrm{TR}$ of the relative generalized trace

$$\mathrm{TR} : \lim_{m \to \infty} HC_0^{\mathrm{cont}}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big) \to HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1)$$

defined in Section 4.1 (26) and the trace map

$$T : HC_0^{\mathrm{cont}}(\mathcal{T}^1, \mathcal{M}^1) \to \mathbb{C}$$

from Lemma 4.1 can be replaced by the trace map

$$T_\infty : \lim_{m \to \infty} HC_0^{\mathrm{cont}}\big(M_m(\mathcal{T}^1), M_m(\mathcal{M}^1)\big) \to \mathbb{C}$$

induced by

$$(S, 0) \mapsto \mathrm{Tr}_\infty(S)$$

This allows us to deduce the equality

$$(T \circ \mathrm{TR} \circ \gamma)(0, \sigma_1 - \sigma_2) = \mathrm{Tr}_\infty\big(\int_0^1 \frac{d\sigma_2}{dt} \cdot \sigma_2^{-1} dt - \int_0^1 \frac{d\sigma_1}{dt} \cdot \sigma_1^{-1} dt\big)$$

proving the lemma. □

Lemma 4.7. *The composition of*

$$\tilde{\gamma} : H_1\big(\mathrm{Coker}(\delta_{\pi_2}(\Delta))\big) \to \mathbb{C}$$

with the exponential $\exp : \mathbb{C} \to \mathbb{C}^*$ *coincides with the Fredholm determinant of the endpoints*

$$(\exp \circ \tilde{\gamma})(\sigma_1, \sigma_2) = \det\big((GL(\pi_1) \circ \sigma_1)(\mathbf{1})^{-1} \cdot (GL(\pi_1) \circ \sigma_2)(\mathbf{1})\big)$$

Here $\pi_1 : \mathcal{T}^1 \to \mathcal{L}(H)$ *is the continuous unital algebra homomorphism given by*

$$\pi_1 : (S, x) \mapsto S \quad \textit{for all} \quad (S, x) \in \mathcal{T}^1$$

Proof. Let there be given an element

$$(\sigma_1, \sigma_2) \in GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big) \times_{GL\left(C_0^2(\Delta^1, \mathcal{M}^1)\right)} GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big)$$

The C^2 map $\sigma_1 \sigma_2^{-1} : \Delta^1 \to GL(\mathcal{T}^1)$ is then of the form

$$\sigma_1 \sigma_2^{-1} = (\alpha, 1) = \big((GL(\pi_1) \circ \sigma_1) \cdot (GL(\pi_1) \circ \sigma_2)^{-1}, 1\big)$$

with $\alpha : \Delta^1 \to \mathcal{G}$ being C^2 and $\alpha(\mathbf{0}) = 1$. It follows that

$$(\exp \circ \tilde{\gamma})(\sigma_1, \sigma_2) = \exp\big(- \mathrm{Tr}_\infty \int_0^1 \frac{d\alpha}{dt} \cdot \alpha^{-1} dt\big)$$

but this is precisely the determinant

$$\det\big(\alpha(\mathbf{1})\big)^{-1} = \det\big((GL(\pi_1) \circ \sigma_1)(\mathbf{1})^{-1} \cdot (GL(\pi_1) \circ \sigma_2)(\mathbf{1})\big)$$

as desired. □

4.3. **Proof of The Main Result.** In this section we draw the consequences of Lemma 4.7, culminating in a proof that the universal determinant invariant equals the universal multiplicative character up to a canonical homomorphism on algebraic K-theory.

Let $\mathrm{Coker}(\delta_{GL(q)})$ and $\mathrm{Coker}(\delta_{GL(\pi_2)})$ denote the cokernel complexes in group homology associated with the group homomorphisms

$$GL(q) : GL\big(\mathcal{L}(H)\big) \to GL\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \quad \text{and} \quad GL(\pi_2) : GL(\mathcal{T}^1) \to GL(\mathcal{M}^1)$$

respectively. Let $I_{GL(q)}$ and $I_{GL(\pi_2)}$ denote their images. See Section 2.1.

The commutative diagram of $\mathbb{C}$-algebra homomorphisms from (20) induces the commutative diagram

$$\begin{tikzcd} GL(\mathcal{T}^1) \arrow[r, "GL(\pi_2)"] \arrow[d, "GL(\pi_1)"'] & I_{GL(\pi_2)} \arrow[d, "GL(R)"] \\ GL\big(\mathcal{L}(H)\big) \arrow[r, "GL(q)"'] & I_{GL(q)} \end{tikzcd}$$

of group homomorphisms. In particular we get the homomorphisms

$$\big(GL(R), GL(\pi_1)\big)_* \quad \text{and} \quad GL(\pi_1)_*$$

making the diagram

$$\begin{tikzcd} H_2(I_{GL(\pi_2)}) \arrow[r, "\partial"] \arrow[d, "GL(R)_*"'] & H_1\big(GL(\mathcal{T}^1), I_{GL(\pi_2)}\big) \arrow[r, "(i\circ\varepsilon)^{-1}"] \arrow[d, "\left(GL(R)\text{,}GL(\pi_1)\right)_*"'] & H_1\big(\mathrm{Coker}(\delta_{GL(\pi_2)})\big) \arrow[d, "GL(\pi_1)_*"] \\ H_2(I_{GL(q)}) \arrow[r, "\partial"'] & H_1\big(GL(\mathcal{L}(H)), I_{GL(q)}\big) \arrow[r, "(i\circ\varepsilon)^{-1}"'] & H_1\big(\mathrm{Coker}(\delta_{GL(q)})\big) \end{tikzcd} \tag{30}$$

commutative. See Section 2.1.

Let $\operatorname{Coker}\big(\delta_{\pi_2}(\Delta)\big)$ denote the cokernel complex in normalized singular homology associated with $\pi_2 : \mathcal{T}^1 \to \mathcal{M}^1$. We then have the commutative diagram

$$\begin{tikzcd} H_2\big(\mathcal{M}_0^1(\Delta)\big) \arrow[r, "\partial"] \arrow[d, "\theta"'] & H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big) \arrow[r, "(i\circ\varepsilon)^{-1}"] \arrow[d, "\theta"'] & H_1\big(\operatorname{Coker}(\delta_{\pi_2}(\Delta))\big) \arrow[d, "\theta"] \\ H_2(I_{GL(\pi_2)}) \arrow[r, "\partial"'] & H_1\big(GL(\mathcal{T}^1), I_{GL(\pi_2)}\big) \arrow[r, "(i\circ\varepsilon)^{-1}"'] & H_1\big(\operatorname{Coker}(\delta_{GL(\pi_2)})\big) \end{tikzcd} \tag{31}$$

Note that

$$i \circ \varepsilon : H_1\big(\operatorname{Coker}(\delta_{\pi_2}(\Delta))\big) \to H_1\big(\mathcal{T}_0^1(\Delta), \mathcal{M}_0^1(\Delta)\big)$$

is an isomorphism by the existence of the continuous linear section of $\pi_2 : \mathcal{T}^1 \to \mathcal{M}^1$

$$s : \mathcal{M}^1 \to \mathcal{T}^1 \qquad x \mapsto (x_{11}, x)$$

See Section 3.1 for details.

Recall that the determinant

$$\det : H_1\big(\operatorname{Coker}(\delta_{GL(q)})\big) \to \mathbb{C}^*$$

is given by

$$(g_1, g_2) \mapsto \det(g_1 g_2^{-1})$$

on generators

$$(g_1, g_2) \in GL\big(\mathcal{L}(H)\big) \times_{GL\left(\mathcal{L}(H)/\mathcal{L}^1(H)\right)} GL\big(\mathcal{L}(H)\big)$$

See Section 2.2 for details.

The content of Lemma 4.7 can now be stated in a homological fashion as the commutativity of the diagram

$$\begin{tikzcd} H_1\big(\operatorname{Coker}(\delta_{\pi_2}(\Delta))\big) \arrow[r, "\theta"] \arrow[d, "\tilde{\gamma}"] & H_1\big(\operatorname{Coker}(\delta_{GL(\pi_2)})\big) \arrow[r, "GL(\pi_1)_*"] & H_1\big(\operatorname{Coker}(\delta_{GL(q)})\big) \arrow[dl, "\det"] \\ \mathbb{C} \arrow[r, "\exp"'] & \mathbb{C}^* & \end{tikzcd} \tag{32}$$

Indeed for each generator

$$(\sigma_1, \sigma_2) \in GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big) \times_{GL\left(C_0^2(\Delta^1, \mathcal{M}^1)\right)} GL\big(C_0^2(\Delta^1, \mathcal{T}^1)\big)$$

we have

$$\begin{aligned} (\det \circ GL(\pi_1)_* \circ \theta)(\sigma_1, \sigma_2) &= \det\big((GL(\pi_1) \circ \sigma_1)(\mathbf{1})^{-1}, (GL(\pi_1) \circ \sigma_2)(\mathbf{1})^{-1}\big) \\ &= \det\big((GL(\pi_1) \circ \sigma_1)(\mathbf{1})^{-1} \cdot (GL(\pi_1) \circ \sigma_2)(\mathbf{1})\big) \\ &= (\exp \circ \tilde{\gamma})(\sigma_1, \sigma_2) \end{aligned}$$

For details see Subsection 3.1.2.

Combining the result of Lemma 4.6 with the commutative diagrams (32), (31) and (30) enables us to calculate as follows

$$\begin{aligned}\exp \circ T \circ \mathrm{TR} \circ \gamma \circ \partial &= \exp \circ \tilde{\gamma} \circ (i \circ \varepsilon)^{-1} \circ \partial \\ &= \det \circ GL(\pi_1)_* \circ \theta \circ (i \circ \varepsilon)^{-1} \circ \partial \\ &= \det \circ GL(\pi_1)_* \circ (i \circ \varepsilon)^{-1} \circ \partial \circ \theta \\ &= \det \circ (i \circ \varepsilon)^{-1} \circ \partial \circ GL(\alpha)_* \circ \theta \\ &= \det \circ \partial \circ GL(R)_* \circ \theta\end{aligned}$$

We now precompose this equation with the relative Hurewicz homomorphism

$$h_2^{\mathrm{rel}} : K_2^{\mathrm{rel}}(\mathcal{M}^1) \to H_2\big(\mathcal{M}_0^1(\Delta)\big)$$

and use the commutativity of the diagram

$$\begin{tikzcd} K_2^{\mathrm{rel}}(\mathcal{M}^1) \arrow[r, "\theta"] \arrow[d, "h_2^{\mathrm{rel}}"'] & K_2(\mathcal{M}^1) \arrow[r, "R_*"] \arrow[d, "h_2"'] & K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \arrow[d, "h_2"] \\ H_2\big(\mathcal{M}_0^1(\Delta)\big) \arrow[r, "\theta"'] & H_2(I_{GL(\pi_2)}) \arrow[r, "GL(R)_*"'] & H_2(I_{GL(q)}) \end{tikzcd}$$

See [18, Paragraphe 6.15-6.18]. We obtain the equation

$$\exp \circ T \circ \mathrm{TR} \circ \gamma \circ \partial \circ h_2^{\mathrm{rel}} = \det \circ \partial \circ GL(R)_* \circ \theta \circ h_2^{\mathrm{rel}} = \det \circ \partial \circ h_2 \circ R_* \circ \theta \tag{33}$$

and proceed with a proof of the main theorem

Theorem 4.8. *The composition of the universal multiplicative character*

$$M_U : K_2(\mathcal{M}^1) \to \mathbb{C}/(2\pi i)\mathbb{Z}$$

with the exponential function

$$\exp : \mathbb{C}/(2\pi i)\mathbb{Z} \to \mathbb{C}^*$$

coincides with the composition of the homomorphism on algebraic K-theory

$$R_* : K_2(\mathcal{M}^1) \to K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

induced by the algebra homomorphism

$$R : \mathcal{M}^1 \to \mathcal{L}(H)/\mathcal{L}^1(H) \qquad x \mapsto q(PxP)$$

and the universal determinant invariant

$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

That is

$$\exp \circ M_U = \mathrm{Det}_U \circ R_*$$

Proof. From the long exact sequence (17) we see that

$$\theta : K_2^{\mathrm{rel}}(\mathcal{M}^1) \to K_2(\mathcal{M}^1)$$

implements the isomorphism

$$\mathrm{Coker}(v) \cong K_2(\mathcal{M}^1)$$

By construction of the universal multiplicative character it is therefore enough to prove the equality

$$\exp\circ\tau_1\circ \mathrm{ch}_2^{\mathrm{rel}} = \mathrm{Det}_U\circ R_*\circ\theta$$

See Section 3.3. But this is immediate from Corollary 4.3, Equation (33) and Definition 2.3

$$\exp\circ\tau_1\circ \mathrm{ch}_2^{\mathrm{rel}} = \exp\circ T\circ \mathrm{TR}\circ\gamma\circ\partial\circ h_2^{\mathrm{rel}} = \det\circ\partial\circ h_2\circ R_*\circ\theta = \mathrm{Det}_U\circ R_*\circ\theta$$

proving the theorem. □

5. Secondary Invariants for Spectral Triples

5.1. **The Multiplicative Character of a Spectral Triple.** In this section we will see that each odd unital q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ with $1 \leq q < 2p$ gives rise to a multiplicative character

$$M_{\mathcal{D}} : K_{2p}(\mathcal{A}) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

in a canonical way. We start by recalling the relevant definitions.

Definition 5.1. *Let $q \in [1, \infty)$. By an odd unital q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ we will understand a separable Hilbert space H a unital $\mathbb{C}$-algebra $\mathcal{A}$ represented on H, and an unbounded self adjoint operator $\mathcal{D}$ such that*

1. *$[\mathcal{D}, a]$ is densely defined and extends to a bounded operator on H for all $a \in \mathcal{A}$.*
2. *$(\lambda - \mathcal{D})^{-1} \in \mathcal{L}^q(H)$ for all $\lambda \in \mathbb{C} - \mathbb{R}$.*

Definition 5.2. *Let $k \in [1, \infty)$. By an odd unital k-summable Fredholm module (H, F) over $\mathcal{A}$ we will understand a separable Hilbert space H, a unital $\mathbb{C}$-algebra $\mathcal{A}$ represented on H, and a bounded self adjoint operator F such that*

1. *$F^2 = 1$*
2. *$[F, a] \in \mathcal{L}^k(H)$ for all $a \in \mathcal{A}$*

Let $p \in \{1, 2, \ldots\}$. In [10, Proposition 1.6] it is clarified that each odd unital $2p$-summable Fredholm module (H, F) over $\mathcal{A}$ gives rise to an algebra homomorphism

$$\rho : \mathcal{A} \to \mathcal{M}^{2p-1}$$

Where $\mathcal{M}^{2p-1}$ is the Banach algebra introduced in Section 3.3. This is essentially due to the observation

$$P\pi(a)(1-P),\ (1-P)\pi(a)P \in \mathcal{L}^{2p}(H)$$

where P is the projection $P = \frac{F+1}{2}$ and $\pi : \mathcal{A} \to \mathcal{L}(H)$ is the representation of $\mathcal{A}$ on H.

By functoriality of algebraic K-theory the algebra homomorphism

$$\rho : \mathcal{A} \to \mathcal{M}^{2p-1}$$

induces a group homomorphism

$$\rho_* : K_{2p}(\mathcal{A}) \to K_{2p}(\mathcal{M}^{2p-1})$$

Composition with the universal multiplicative character

$$M_U : K_{2p}(\mathcal{M}^{2p-1}) \to \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

thus yields a homomorphism

$$M_U \circ \rho_* : K_{2p}(\mathcal{A}) \to \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

This is the multiplicative character of the odd unital $2p$-summable Fredholm module (H, F) over $\mathcal{A}$. See [10].

Let $q \in [1, \infty)$. Let $(\mathcal{A}, H, \mathcal{D})$ be an odd unital q-dimensional spectral triple with representation

$$\pi : \mathcal{A} \to \mathcal{L}(H)$$

Let P be the projection onto the positive part of the spectrum of $\mathcal{D}$.

$$P = 1_{[0,\infty)}(\mathcal{D})$$

By [11, Corollary 2.5] the commutator $[2P - 1, \pi(a)]$ is in $\mathcal{L}^k(H)$ for all $k > q$. In particular the couple $(2P - 1, H)$ is an odd k-summable Fredholm module over $\mathcal{A}$.

Summarizing we get a multiplicative character

$$M_{\mathcal{D}} : K_{2p}(\mathcal{A}) \to \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

associated with each odd unital q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ with $1 \leq q < 2p$.

5.2. **The Determinant Invariant of a Spectral Triple.** In this section given an odd unital q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ with $1 \leq q < 2$ we construct a canonical exact sequence of $\mathbb{C}$-algebras

$$X_{\mathcal{D}} : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{E} \xrightarrow{s} \mathcal{B} \longrightarrow 0$$

and a surjective algebra homorphism $R : \mathcal{A} \to \mathcal{B}$. By consequence we get a determinant invariant

$$\mathrm{Det}_{X_{\mathcal{D}}} : K_2(\mathcal{B}) \to \mathbb{C}^*$$

and a group homomorphism

$$R_* : K_2(\mathcal{A}) \to K_2(\mathcal{B})$$

We then prove the equality

$$\mathrm{Det}_{X_{\mathcal{D}}} \circ R_* = \exp \circ M_{\mathcal{D}}$$

thus giving a refinement of Theorem 4.8.

Let $(\mathcal{A}, H, \mathcal{D})$ be an odd unital q-dimensional spectral triple with $1 \leq q < 2$. Let

$$\pi : \mathcal{A} \to \mathcal{L}(H)$$

denote the representation of $\mathcal{A}$ on the separable Hilbert space H. Let P denote the projection onto the positive part of the spectrum of $\mathcal{D}$

$$P = 1_{[0,\infty)}(\mathcal{D})$$

Let H^+ denote the Hilbert space PH. Recall that $[P, \pi(a)] \in \mathcal{L}^2(H)$ for all $a \in \mathcal{A}$ by [11, Corollary 2.5].

We define a $\mathbb{C}$-subalgebra $\mathcal{E}$ of $\mathcal{L}(H^+)$ by

$$\mathcal{E} = \{P\pi(a)P + T \mid T \in \mathcal{L}^1(H^+) \text{ and } a \in \mathcal{A}\}$$

Clearly $\mathcal{E}$ contains the operators of trace class and is closed under scalar multiplication and addition. To show that $\mathcal{E}$ is closed under multiplication we calculate as follows

$$\begin{aligned}
&(P\pi(a)P+T)(P\pi(b)P+S)\\
&\quad = P\pi(a)P\pi(b)P+TS+TP\pi(b)P+P\pi(a)PS\\
&\quad = P[\pi(a),P][\pi(b),P]P+P\pi(ab)P+TS+TP\pi(b)P+P\pi(a)PS\\
&\quad \in \mathcal{E}
\end{aligned}$$

We define the $\mathbb{C}$-algebra $\mathcal{B}$ to be the quotient of $\mathcal{E}$ by $\mathcal{L}^1(H^+)$.

By construction we have an exact sequence

$$X_{\mathcal{D}} : 0 \longrightarrow \mathcal{L}^1(H^+) \xrightarrow{\;i\;} \mathcal{E} \xrightarrow{\;s\;} \mathcal{B} \longrightarrow 0$$

Together with the inclusion $\iota : \mathcal{E} \to \mathcal{L}(H^+)$ this is the data needed to define a determinant invariant

$$\mathrm{Det}_{X_{\mathcal{D}}} : K_2(\mathcal{B}) \to \mathbb{C}^* \qquad \mathrm{Det}_{X_{\mathcal{D}}} = \mathrm{Det}_U \circ \iota_*$$

See Section 2.2. This is the determinant invariant of the spectral triple $(\mathcal{A}, H, \mathcal{D})$.

We define a surjective algebra homomorphism

$$R : \mathcal{A} \to \mathcal{B}$$

by the assignment

$$R(a) = q\big(P\pi(a)P\big)$$

We are now ready to prove the following refinement of our main result

Theorem 5.3. *The diagram*

$$\begin{array}{ccc}
K_2(\mathcal{A}) & \xrightarrow{R_*} & K_2(\mathcal{B})\\
{\scriptstyle M_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Det}_{X_{\mathcal{D}}}}\\
\mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow[\exp]{} & \mathbb{C}^*
\end{array}$$

is commutative. In particular the multiplicative character and the determinant invariant associated with an odd unital q-dimensional spectral triple $(\mathcal{A}, H, \mathcal{D})$ *with* $1 \leq q < 2$ *coincides up to the homomorphism*

$$R_* : K_2(\mathcal{A}) \to K_2(\mathcal{B})$$

Proof. Let $\iota : \mathcal{B} \to \mathcal{L}(H^+)/\mathcal{L}^1(H^+)$ denote the algebra homomorphism induced by the inclusion $\iota : \mathcal{E} \to \mathcal{L}(H^+)$. We then have the commutative diagram

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\quad R \quad} & \mathcal{B}\\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \iota}\\
\mathcal{M}^1 & \xrightarrow[R]{} & \mathcal{L}(H^+)/\mathcal{L}^1(H^+)
\end{array}$$

Together with the result of Theorem 4.8 this allows us to calculate as follows

$$\exp \circ M_{\mathcal{D}} = \exp \circ M_U \circ \rho_* = \mathrm{Det}_U \circ R_* \circ \rho_* = \mathrm{Det}_U \circ \iota_* \circ R_* = \mathrm{Det}_{X_{\mathcal{D}}} \circ R_*$$

proving the theorem. $\square$

6. Calculations of the Multiplicative Character

In this section we study the homomorphism

$$M_{-i\frac{d}{dt}} : K_2\big(C^\infty(S^1)\big) \to \mathbb{C}/(2\pi i)\mathbb{Z}$$

induced by the spectral triple $\big(C^\infty(S^1), L^2(S^1), -i\frac{d}{dt}\big)$. In particular we are interested in calculating the quantity

$$M_{-i\frac{d}{dt}}\{\alpha,\beta\} \in \mathbb{C}/(2\pi i)\mathbb{Z}$$

Here $\alpha,\beta : S^1 \to \mathbb{C}^*$ are two smooth loops and $\{\alpha,\beta\} \in K_2\big(C^\infty(S^1)\big)$ denotes their Steinberg symbol. See [27] for additional information.

For details on the construction of the spectral triple $\big(C^\infty(S^1), L^2(S^1), -i\frac{d}{dt}\big)$ and the associated Fredholm module $\big(F, L^2(S^1)\big)$ we refer to [11, Appendix C] for example. For our purposes it is sufficient to know the form of the corresponding algebra homomorphism

$$\rho : C^\infty(S^1) \to \mathcal{M}^1 \subseteq \mathcal{L}(H \oplus H)$$

This map is induced by the assignment

$$\rho(z) = \begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix} \quad \text{and} \quad \rho(z^{-1}) = \begin{pmatrix} S^* & 0 \\ 1 - SS^* & S \end{pmatrix}$$

for the generating loops z and z^{-1}. Here $S : H \to H$ is the shift operator given by

$$S(e_n) = e_{n+1} \quad \text{for all} \quad n \in \{0,1,2,\ldots\}$$

where (e_n) is an orthonormal basis for the Hilbert space H indexed by the positive integers. Thus by definition we have

$$M_{-i\frac{d}{dt}} = M_U \circ \rho_*$$

where $\rho_* : K_2\big(C^\infty(S^1)\big) \to K_2(\mathcal{M}^1)$ is induced by ρ and $M_U : K_2(\mathcal{M}^1) \to \mathbb{C}/(2\pi i)\mathbb{Z}$ is the universal multiplicative character.

Now, let $\alpha,\beta : S^1 \to \mathbb{C}^*$ be two smooth loops. Let n and m be the winding numbers of α and β respectively. We can then find smooth loops $a : S^1 \to \mathbb{C}$ and $b : S^1 \to \mathbb{C}$ such that

$$e^{a(z)} = z^{-n}\alpha \quad \text{and } e^{b(z)} = z^{-m}\beta$$

for all $z \in S^1$. It follows that

$$\alpha(z) = z^n e^{a(z)} \quad \text{and } \beta(z) = z^m e^{b(z)}$$

Let $a(S), b(S) \in \mathcal{L}(H)$ denote the operators

$$a(S) = \textstyle\sum_{k\geq 0} a_k S^k + \sum_{k>0} a_{-k}(S^k)^* \quad \text{and} \quad b(S) = \sum_{l\geq 0} b_l S^l + \sum_{l>0} b_{-l}(S^l)^* \tag{34}$$

Here $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and $b(z) = \sum_{l\in\mathbb{Z}} b_l z^l$ are the Fourier expansions of $a : S^1 \to \mathbb{C}$ and $b : S^1 \to \mathbb{C}$.

By the general theory of Toeplitz operators we have

$$q\big(P\rho(e^a)P\big) = q(e^{a(S)}) \quad \text{and} \quad q\big(P\rho(e^b)P\big) = q(e^{b(S)}) \tag{35}$$

where $q : \mathcal{L}(H) \to \mathcal{L}(H)/\mathcal{L}^1(H)$ is the quotient map and P is the projection

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H)$$

Indeed one could start by proving the equalities

$$q\big(P\rho(e^a)P\big) = q\big(e^{P\rho(a)P}\big) \quad \text{and} \quad q\big(P\rho(a)P\big) = q\big(a(S)\big)$$

and then deduce from the latter one that also $q(e^{P\rho(a)P}) = q(e^{a(S)})$.

In order to calculate the multiplicative character of the Steinberg symbol $\{\alpha, \beta\} \in K_2\big(C^\infty(S^1)\big)$ we need a lemma on Fredholm determinants.

Lemma 6.1. *Suppose that $x, y \in \mathcal{L}(H)$ satisfies*

$$x - y \in \mathcal{L}^1(H)$$

The operator $e^x e^{-y} \in \mathcal{G}$ is then of determinant class and the determinant is given by

$$\det(e^x e^{-y}) = e^{\mathrm{Tr}(x-y)}$$

Proof. By Duhamel's formula we have

$$e^x e^{-y} = 1 + \int_0^1 e^{tx}(x-y)e^{-ty}dt$$

proving that $e^x e^{-y} \in \mathcal{G}$.

Let $F : [0,1] \to \mathcal{G}$ denote the smooth path

$$F : t \mapsto e^{tx}e^{-ty}$$

Let $f : [0,1] \to \mathbb{C}^*$ denote the smooth path

$$f : t \mapsto \det\big(F(t)\big)$$

By the Gohberg-Krein Lemma, [31, Chapter I.9], we have

$$f^{-1}\frac{df}{dt} = \mathrm{Tr}\big(F^{-1}\frac{dF}{dt}\big)$$

The derivative of F is given by

$$\frac{dF}{dt} = e^{tx}(x-y)e^{-ty}$$

The operator trace of $F^{-1}\frac{dF}{dt}$ is therefore

$$\mathrm{Tr}\big(F^{-1}\frac{dF}{dt}\big) = \mathrm{Tr}(x-y)$$

Since $f(0) = 1$ and

$$\frac{df}{dt} = \mathrm{Tr}(x-y) \cdot f$$

we must have $f(t) = e^{t\mathrm{Tr}(x-y)}$ proving the Lemma. □

We are now ready to do our calculation of the multiplicative character.

Theorem 6.2. *The multiplicative character associated with the spectral triple*

$$\big(C^\infty(S^1), L^2(S^1), -i\frac{d}{dt}\big)$$

has the concrete expression

$$\begin{aligned}&(\exp \circ M_U \circ \rho_*)\{\alpha,\beta\} \\ &\quad = \mathrm{Det}_U\{q(S)^n q(e^{a(S)}), q(S)^m q(e^{b(S)})\} \\ &\quad = (-1)^{nm} \exp\big(n\mathrm{Tr}[Sb(S),S^*] - m\mathrm{Tr}[Sa(S),S^*] + \mathrm{Tr}[a(S),b(S)]\big)\end{aligned} \tag{36}$$

when applied to the Steinberg symbol

$$\{\alpha,\beta\} \in K_2\big(C^\infty(S^1)\big)$$

Proof. The first equality

$$(\exp \circ M_U \circ \rho_*)\{\alpha,\beta\} = \mathrm{Det}_U\{q(S)^n q(e^{a(S)}), q(S)^m q(e^{b(S)})\}$$

is immediate from Theorem 4.8 and Equation (35).

The skew-symmetry and bilinearity of the Steinberg symbol yield the identity

$$\begin{aligned}&\mathrm{Det}_U\{q(S)^n q(e^{a(S)}), q(S)^m q(e^{b(S)})\} \\ &\quad = \mathrm{Det}_U\big(\{q(S),q(S)\}^{nm}\{q(S), q(e^{nb(S)-ma(S)})\}\{q(e^{a(S)}), q(e^{b(S)})\}\big)\end{aligned}$$

See [27]. Since the determinant invariant

$$\mathrm{Det}_U : K_2\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to \mathbb{C}^*$$

respects the group structure we get

$$\begin{aligned}&\mathrm{Det}_U\{q(S)^n q(e^{a(S)}), q(S)^m q(e^{b(S)})\} \\ &\quad = \mathrm{Det}_U\{q(S),q(S)\}^{nm} \cdot \mathrm{Det}_U\{q(S), q(e^{nb(S)-ma(S)})\} \cdot \mathrm{Det}_U\{q(e^{a(S)}), q(e^{b(S)})\}\end{aligned}$$

By [5] and Section 2.3 we have the equalities

$$\mathrm{Det}_U\{q(S),q(S)\} = -1 \quad \text{and} \quad \mathrm{Det}_U\{q(e^{a(S)}), q(e^{b(S)})\} = e^{\mathrm{Tr}[a(S),b(S)]}$$

We will therefore only concentrate on the quantity

$$\mathrm{Det}_U\{q(S), q(e^{nb(S)-ma(S)})\} \in \mathbb{C}^*$$

By [16, Lemme 4.2] the Hurewicz homomorphism has the expression

$$\begin{aligned}&h_2\{q(S), q(e^{nb(S)-ma(S)})\} \\ &\quad = \big(d_{13}\big(q(e^{nb(S)-ma(S)})\big), d_{12}\big(q(S)\big)\big) - \big(d_{12}\big(q(S)\big), d_{13}\big(q(e^{nb(S)-ma(S)})\big)\big) \\ &\qquad \in H_2(I_{GL(q)})\end{aligned}$$

when applied to the Steinberg symbol $\{q(S), q(e^{nb(S)-ma(S)})\} \in K_2\big(C^\infty(S^1)\big)$. Here the maps

$$d_{12}, d_{13} : GL_m\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big) \to GL_{3m}\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

are given by

$$d_{12}(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d_{13}(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{pmatrix}$$

for each $x \in GL_m\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$

Now, define the invertible matrices $L_S \in GL_3\big(\mathcal{L}(H)\big)$ and $L_{e^{nb(S)-ma(S)}} \in GL_3\big(\mathcal{L}(H)\big)$ by

$$L_S = \begin{pmatrix} S & 1-SS^* & 0 \\ 0 & S^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_{e^{nb(S)-ma(S)}} = \begin{pmatrix} e^{nb(S)-ma(S)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{ma(S)-nb(S)} \end{pmatrix}$$

We then have

$$GL(q)(L_S) = d_{12}\big(q(S)\big) \quad \text{and} \quad GL(q)(L_{e^{nb(S)-ma(S)}}) = d_{13}\big(q(e^{nb(S)-ma(S)})\big)$$

It follows that the cycle

$$(0, L_S \cdot L_{e^{nb(S)-ma(S)}} - L_{e^{nb(S)-ma(S)}} \cdot L_S) \in \mathrm{Tot}_1\big(\mathrm{Cone}(GL(q))[-1]\big)$$

represents the element

$$(\partial \circ h_2)\{q(S), q(e^{nb(S)-ma(S)})\} \in H_1\big(GL\big(\mathcal{L}(H)\big), I_{GL(q)}\big)$$

Here the boundary map

$$\partial : H_2(I_{GL(q)}) \to H_1\big(GL\big(\mathcal{L}(H)\big), I_{GL(q)}\big)$$

is associated with the surjective group homomorphism

$$GL(q) : GL\big(\mathcal{L}(H)\big) \to I_{GL(q)} \subseteq GL\big(\mathcal{L}(H)/\mathcal{L}^1(H)\big)$$

Under the isomorphisms

$$H_1\big(GL\big(\mathcal{L}(H)\big), I_{GL(q)}\big) \xrightarrow{(i\circ\varepsilon)^{-1}} H_1\big(\mathrm{Coker}(\delta_{GL(q)})\big) \xrightarrow{\psi} K_{GL(q)}/\Gamma_{GL(q)}$$

from Theorem 2.1 and Theorem 2.2 the class of the cycle

$$(0, L_S \cdot L_{e^{nb(S)-ma(S)}} - L_{e^{nb(S)-ma(S)}} \cdot L_S) \in \mathrm{Tot}_1\big(\mathrm{Cone}(GL(q))[-1]\big)$$

becomes

$$\begin{aligned} & L_S \cdot L_{e^{nb(S)-ma(S)}} (L_S)^{-1} L_{e^{ma(S)-nb(S)}} \\ & \quad = \begin{pmatrix} Se^{nb(S)-ma(S)} & 1-SS^* \\ 0 & S^* \end{pmatrix} \cdot \begin{pmatrix} S^* e^{ma(S)-nb(S)} & 0 \\ (1-SS^*)e^{ma(S)-nb(S)} & S \end{pmatrix} \\ & \quad = Se^{nb(S)-ma(S)} S^* e^{ma(S)-nb(S)} + (1-SS^*)e^{ma(S)-nb(S)} \\ & \qquad \in K_{GL(q)}/\Gamma_{GL(q)} \end{aligned}$$

The determinant invariant of the symbol is therefore given by

$$\mathrm{Det}_U\{q(S), q(e^{nb(S)-ma(S)})\} = \det\big(Se^{nb(S)-ma(S)} S^* e^{ma(S)-nb(S)} + (1-SS^*)e^{ma(S)-nb(S)}\big)$$

In order to proceed we use the identities

$$\begin{aligned} &Se^{nb(S)-ma(S)}S^*e^{ma(S)-nb(S)} + (1-SS^*)e^{ma(S)-nb(S)} \\ &\quad = e^{S\big(nb(S)-ma(S)\big)S^*}SS^*e^{ma(S)-nb(S)} + (1-SS^*)e^{ma(S)-nb(S)} \\ &\quad = e^{S\big(nb(S)-ma(S)\big)S^*}e^{ma(S)-nb(S)} \end{aligned}$$

Since the operator $S\big(nb(S)-ma(S)\big)S^*-\big(nb(S)-ma(S)\big)$ can be expressed using a commutator

$$S\big(nb(S)-ma(S)\big)S^* - \big(nb(S)-ma(S)\big) = \big[S\big(nb(S)-ma(S)\big), S^*\big]$$

it must be of trace class. By Lemma 6.1 we conclude that

$$\begin{aligned} &\det\big(Se^{nb(S)-ma(S)}S^*e^{ma(S)-nb(S)} + (1-SS^*)e^{ma(S)-nb(S)}\big) \\ &\quad = \exp\Big(n\mathrm{Tr}\big(Sb(S)S^*-b(S)\big) - m\mathrm{Tr}\big(Sa(S)S^*-a(S)\big)\Big) \end{aligned}$$

proving the Theorem. □

We can evolve slightly on the above result

Corollary 6.3. *The exponent in expression* (36) *satisfies*

$$\begin{aligned} &n\mathrm{Tr}[Sb(S),S^*] - m\mathrm{Tr}[Sa(S),S^*] + \mathrm{Tr}[a(S),b(S)] \\ &\quad = ma_0 - nb_0 + \sum_{k\in\mathbb{Z}} ka_{-k}b_k \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} ma(\theta)-nb(\theta)\,d\theta + \frac{1}{2\pi i}\int_0^{2\pi} a(\theta)b'(\theta)\,d\theta \end{aligned}$$

Thus we have the integral formula

$$(\exp\circ M_U\circ\rho_*)\{\alpha,\beta\} = (-1)^{nm}\exp\Big(\frac{1}{2\pi}\int_0^{2\pi} ma(\theta)-nb(\theta)\,d\theta + \frac{1}{2\pi i}\int_0^{2\pi} a(\theta)b'(\theta)\,d\theta\Big)$$

for the multiplicative character.

Proof. The identities

$$\mathrm{Tr}[a(S),b(S)] = \sum_{k\in\mathbb{Z}} ka_{-k}b_k = \frac{1}{2\pi i}\int_0^{2\pi} a(\theta)b'(\theta)d\theta$$

have been proved during the example on page 150-152 of [15].

To show that

$$n\mathrm{Tr}[Sb(S),S^*] = -nb_0$$

we use that

$$Sb(S)S^* - b(S) = -\sum_{n\geq 0} b_nS^n(1-SS^*) - \sum_{n<0} b_n(1-SS^*)(S^*)^{-n}$$

Since the trace of the quantities

$$S^n(1-SS^*) \quad \text{and} \quad (1-SS^*)(S^*)^n$$

vanish for any $n > 0$ we get

$$\mathrm{Tr}\big(Sb(S)S^* - b(S)\big) = -b_0$$

proving the corollary. □

References

[1] S. Baaj, P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens*, C. R. Acad. Sci. Paris **296**, (1983), 139–149.
[2] B. Blackadar, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, Springer-Verlag New York, 1986.
[3] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Math., vol. 87, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
[4] L. G. Brown, *The Determinant Invariant for Operators with Trace Class Self Commutators*, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin, Heidelberg, and New York, 1973, 210-228.
[5] L. G. Brown, *Operator Algebras and Algebraic K.theory*, Bull. Amer. Math. Soc., **81**, 1973, 1119-1121.
[6] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Mathematical Series, vol. 19, Princeton Univ. Press, Princeton, 1956.
[7] A. Connes, *Non Commutative Differential Geometry*, Publ. math. IHES No. 62, 1985, 257-360.
[8] A. Connes, *Cohomologie Cyclique et Foncteurs* Ext^n, C. R. Acad. Sci. Paris **296**, 1983, 953-959.
[9] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
[10] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, K-theory **2**, 1988, 431-463.
[11] A. L. Carey and J. Phillips, *Unbounded Fredholm Modules and Spectral Flow*, Can. J. Math. Vol. **50** (1998), 673–718.
[12] A. L. Carey, J. Phillips and F. A. Sukochev, *On Unbounded p-summable Fredholm Modules*, Advances in Mathematics. Vol. **151** Issue **2**, 2000, 140-163.
[13] J. L. Dupont, *Curvature and Characteristic Classes*, Lecture Notes in Mathematics, vol. 640, Springer Verlag, Berlin, Heidelberg, New York, 1978.
[14] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
[15] J. W. Helton and R. E. Howe, *Integral Operators: Commutators, Traces, Index, and Homology*, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin, Heidelberg, New York, 1973, 141-209.
[16] G. Ginot, *Formules explicites pour le caractère de Chern en K-théorie algébrique*, Ann. Inst. Fourier, Grenoble **54**, 2004, 2327-2355.
[17] A. Grothendieck, *Produits Tensoriels Topologiques*, Mem. AMS., No. 16, 1955.
[18] M. Karoubi, *Homologie Cyclique et K-théorie*, Astérisque No. 149, 1987.
[19] M. Levine, *Relative Milnor K-Theory*, K-theory **6**, 1992, 113-175.
[20] J. -L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Scient. Ecole Norm. Sup. **9**, 1976, 309-377.
[21] J. -L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, Springer-Verlag, New York, Berlin, Heidelberg, 1992.
[22] J. -L. Loday and D. Quillen, *Cyclic Homology and The Lie Algebra Homology of Matrices*, Comment. Math. Helv. **59**, 1984, 565-591.
[23] D. Macduff and P. de la Harpe, *Acyclic groups of automorphisms*, Comment. Math. Helv. **58**, 1983, 48-71.
[24] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, Academic Press, London, New York, San Francisco, 1979.
[25] D. Quillen, *Higher Algebraic K-theory I*, Algebraic K-theory I (Battelle, 1972) (H. Bass, ed.), Lecture Notes in Math., vol. 341, Springer-Verlag, New York, Berlin, Heidelberg, 1973, 85-147.
[26] J. Rosenberg, *Algebraic K-theory and Its Applications*, Graduate Texts in Math., vol. 147, Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[27] J. Rosenberg, *Comparison Between Algebraic and Topological K-Theory for Banach Algebras and C^*-Algebras*, Handbook of Algebraic K-theory, Springer, 2004, 843-874.
[28] A. Presley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.
[29] B. Simon *Trace Ideals and Their Applications*, Cambridge University Press, 1979.
[30] U. Tillmann, *Factorization of the (Relative) Chern Character through Lie Algebra Homology*, K-theory **6**, 1992, 457-463.
[31] A. Wassermann, *Analysis of Operators*, Michaelmas, 2006.
[32] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math., vol. 61, Springer-Verlag, New York, Heidelberg, Berlin, 1978.